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Optimisation de forme : systèmes nonlinéaires et mécanique des fluides

Shape Optimization: Nonlinear Systems and Fluid Mechanics

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Résumé : Le contrôle par rapport à la géométrie de systèmes fluides est une activité récente qui vient en complément des contrôles plus classiques par rapport au champ de force volumique ou par injection/aspiration de fluide. Nous développons dans cette thèse l'application des méthodes d'optimisation de forme au système de Navier-Stokes qui régit les écoulements des fluides newtoniens.

L'analyse du système stationnaire est menée en insistant en particulier sur deux points : tout d'abord la recherche d'un affaiblissement des hypothèses permettant d'assurer l'existence d'une variation au premier ordre des champs de vitesse et de pression par rapport à une perturbation de la géométrie ; l'hypothèse classique, qui consiste à supposer la viscosité suffisament élevée, est en effet trop restrictive pour la plupart des applications. Ensuite la preuve de l'existence et la détermination effective du gradient de forme d'une classe de fonctionnelles "de bord", nécessitant l'évaluation d'intégrales surfaciques sur la frontière du fluide. Le rapport portance/trainée fait par example partie de ces fonctionnelles qui requièrent une analyse approfondie de la régularité des écoulements.

Pour une viscosité trop faible, la capacité prédictive du modèle stationnaire devient toutefois trop aléatoire. Pour cette raison, nous complétons l'étude précédente par l'analyse de la sensibilité par rapport à la géométrie du système dynamique de Navier-Stokes et mettons en évidence l'existence d'une variation du premier ordre du champ de vitesse associé au système lorsque le domaine spatio-temporel qui contient le fluide est soumis à une perturbation.

Deux parties supplémentaires de portée plus générale complètent ces études. La première est un prolongement du premier chapitre visant à affaiblir encore les hypothèses nécessaires à l'obtention de la dérivabilité de l'état de systèmes nonlinéaires par rapport à la géométrie. L'étude porte précisement sur l'établissement de résultats de différentiablité générique, c'est-à-dire valables pour "presque toute géométrie". Le système concerné est une équation quasilinéaire elliptique scalaire, plus simple que le modèle de Navier-Stokes mais présentant de nombreuses analogies structurelles.

Enfin, une dernière partie contribue au rapprochement de deux méthodes d'optimisation de forme utilisées dans ce document : la méthode des vitesses et la méthode de perturbation de l'identité. On montre l'équivalence des deux concepts de continuité par rapport au domaine propres à chacun de ces cadres.

Abstract: The control with respect to the geometry of the fluid flows is complementary to the more classical control with respect to the force field or boundary values. We investigate in this thesis the application of shape optimization methods to the Navier-Stokes system that governs the behavior of newtonian fluids.

We analyse the stationary system with a special emphasis on two points: at first, the weakening of the "high-viscosity" assumption required to prove the existence of a shape derivative of the velocity and pressure fields, assumption which is unrealistic for most applications. After that, the existence and explicit calculation of a class of boundary functionals (including the lift to drag ratio) whose study requires a deep analysis of the regularity of the flow.

Under a given viscosity threshold, the stationary model becomes unaccurate. Therefore, we analyse also the dynamical Navier-Stokes model with moving boundaries and the influence of geometric perturbation on the flow.

The two other parts of this thesis have a broader scope. The first one is an extension of the study of shape differentiablity for nonlinear systems. We show what conditions ensure the generic regularity with respect to the shape of the solutions of a scalar elliptic quasilinear equation, that is the existence for "almost every geometry" of their shape derivatives.

The last part is a contribution to the comparison of two transformation methods used in this document: the velocity method and the perturbation of the identity method. We prove the equivalence of the definitions of continuity of a shape-dependent function that exist in both frameworks.

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Sébastien BOISGÉRAULT

Contents

Introduction Générale

Motivation

A-t'on encore besoin de présenter le challenge que constitue l'étude des problèmes de la Mécanique des Fluides et les bénéfices que peut en retirer l'aéronautique ? L'utilisation systématique de méthodes mathématiques et numériques pour améliorer les performances de systèmes fluides est encore limitée par leur grande complexité. Toutefois, l'évolution de la puissance des moyens de calculs permet désormais d'envisager des simulations de dispositifs complexes qui semblaient hors d'atteinte il y a quelques années encore. La recherche théorique associée se trouve dynamisée et enrichie par cette possibilité de tester effectivement certains schémas théoriques, par la nécessité de faire face aux nouveaux problèmes issus du développement des simulations.

Dans cette discipline, le modèle mathématique prédominant, qui présente sans doute le meilleur compromis entre simplicité et capacité prédictive, est le système de Navier-Stokes qui gouverne l'évolution des fluides newtoniens. L'étude mathématique de ce système a mobilisé une communauté scientifique nombreuse depuis l'article séminal de Jean Leray de 1934 *Sur le mouvement d'un fluide visqueux emplissant l'espace*. La théorie qui en résulte n'est pas encore complète : par exemple, la question de savoir si ce modèle est ou non intrinsèquement turbulent (de façon plus précise, de savoir si des données initiales et des forces volumiques régulières génèrent systématiquement des écoulements réguliers) est encore ouverte. Cette question fondamentale a récemment dépassé l'audience des mathématiciens et des physiciens avec la médiatisation des 7 *Millenium Prize Problems* décrits par le Clay Mathematical Institute dont le problème évoqué plus haut fait partie. Les 7 millions de dollars américains associés à la résolution de ces problèmes ne sont sans doute pas étrangers à ce soudain intérêt du grand public ...

L'étude de la sensibilité par rapport à la géométrie du système des équations de Navier-Stokes fait partie des thèmes relativement récents et prometteurs de l'optimisation et du contrôle des équations aux dérivées partielles. Ils complètent les analyses "classiques" qui visent à améliorer les caractéristiques d'un écoulement en jouant sur les forces volumiques appliquées sur le fluide ou les conditions de flux sur la frontière.

Par *optimisation*, il est fait référence à l'étude du problème de Navier-Stokes stationnaire par rapport aux données géométriques, qui est réalisée dans le premier chapitre. L'analyse du problème dynamique, c'est-à-dire l'aspect *contrôle* par rapport à la géométrie est l'objet du dernier chapitre. Les deux chapitres intermédiaires sont complémentaires. Le deuxième chapitre s'emploie à rapprocher deux méthodes parentes de l'optimisation de forme, méthodes qui sont toutes deux utilisées dans cette thèse ; ses résultats ont donc une portée plus vaste que le seul cadre de l'analyse géométrique de l'équation de Navier-Stokes. Enfin le troisème chapitre constitue un prolongement de certaines questions soulevées par le premier chapitre. Les équations de Navier-Stokes y sont abandonnées au profit des équations quasilinéaires, plus simples à analyser mais similaires à de nombreux points de vue.

Une description plus détaillée du contenu de ces chapitres ainsi que leur position par rapports aux travaux déja existant est proposée dans la section suivante de cette introduction générale. Une rappel plus bref est réalisé en début de chaque chapitre.

Contexte de l'étude

1 - Les équations de Navier-Stokes : analyse de forme du cas stationnaire

Les méthodes nécessaires à l'analyse de sensibilité par rapport à la géométrie des équations de Navier-Stokes sont essentiellement issues de la même analyse menée sur les systèmes elliptiques (cf. [47] par exemple). Le passage intermédiaire entre l'étude de ces systèmes et des équations de Navier-Stokes est naturellement l'étude de Navier-Stokes linéarisé autour du repos, c'est-à-dire du système de Stokes, effectuée dans [45].

L'étude du système de Navier-Stokes nonlinéaire a été ensuite réalisée dans le cadre de la méthode de perturbation de l'identité dans [6]. L'étude mène à la fois à un résultat de différentiabilité par rapport à la géométrie de l'état (c'est-à-dire le couple des champs de vitesse et de pression) et à une détermination du gradient de forme de la trainée.

Le chapitre 1 de cette thèse s'inscrit dans une démarche tout à fait analogue à ces documents. On les prolonge dans plusieurs directions. On affaiblit tout d'abord l'hypothèse de type "haute viscosité" qui est requise dans [6]. Elle est remplacée par une hypothèse de non-singularité du système de Navier-Stokes. En abandonnant l'hypothèse de haute viscosité on peut également perdre l'unicité de la solution du système, mais l'hypothèse qui la remplace assure tout de même une unicité *locale* de la solution. On est donc conduit à une analyse de sensibilité de *branches de solutions*.

Le deuxième effort a porté sur le renforcement des résultats de différentiabilité de l'état par rapport à la géométrie. On montre comment la connaissance de la régularité des données peut être répercutée pour obtenir l'existence de la dérivée de forme des champs de vitesse et de pression dans des espaces fonctionnels de régularité la plus élevée possible.

L'intérêt de ce type de résultats est de permettre de montrer la différentiabilité par rapport au domaine de certaines fonctionnelles qui ne sont bien définies qu'en présence d'écoulements réguliers. La régularité "élémentaire" (champs de pression, de vitesse et du gradient de la vitesse de carré sommable) obtenue naturellement comme résultat de la formulation faible du système de Navier-Stokes, ne permet par exemple pas de considérer des fonctionnelles aussi importantes que le rapport portance/trainée associé à l'écoulement autour d'une aile d'avion. Sur la base des résultats de sensibilité que nous avons établi, nous développons les étapes du calcul de forme nécessaires à l'évaluation du gradient de forme de telles fonctionnelles, qui fondamentalement ne dépendent que de la valeur normale du tenseur de stress au niveau de l'interface fluide-solide. Nous effectuons également la même démarche sur une classe de fonctionnelles jamais étudiées à notre connaissance du point de vue du calcul de forme : les fonctionnelles lagrangiennes dont la valeur est le résultat d'une intégration spatiale et temporelle le long des trajectoires que décrivent les particules fluides.

2 - Topologies sur les formes. Comparaison des méthodes des vitesses et de perturbation de l'identité

Parmi les diverses thématiques de l'optimisation de forme, les méthodes de transformations (appelées également méthodes de perturbation) ont acquis une grande importance en raison de leur large spectre d'application et du caractère systématique du calcul de forme qui leur est associé. A la base de ces méthodes, on trouve le même système de paramétrisation des géométries : un domaine de référence de $\mathbb{R}^n$ étant déterminé, les formes perturbées sont générées comme image de ce domaine par une transformation définie sur un sous-ensemble de $\mathbb{R}^n$ contenant le domaine de référence.

Dans ce cadre, la plupart des analyses réalisées se réclament de l'un ou l'autre des deux formalismes essentiels suivants : la méthode de perturbation de l'identité ([36]) et la méthode des vitesses ([47], [20]).

Dans le premier de ces formalismes, on travaille directement sur la transformation θ qui fait correspondre au domaine de référence Ω le domaine perturbé Ω_θ ou bien de façon équvalente sur l'écart $u = \theta - I$ de cette transformation à l'identité. Dans le second, les perturbations sont induites par un champ de vitesse V qui définit un unique flot $T_t(V)$, transformation définie sur un domaine $\overline{D}$ appelé région de conception. Ce flot permet de définir une famille Ω_t de domaines par la formule $\Omega_t = T_t(V)(\Omega)$.

Les efforts de rapprochement entre les deux méthodes ont surtout porté sur la caractérisation des familles θ_t de transformations de $\overline{D}$ qui pouvaient être générées. Dans [47], on détermine des conditions nécessaires et suffisantes sous lesquelles une famille de transformation $\mathcal{C}^k$ à un paramètre définie sur un domaine régulier par morceaux peut être générée comme flot associé à un champ de vitesse. Cette démarche est généralisée aux tranformations lipschitziennes et aux régions de conceptions ouvertes quelconques dans [18].

Le calcul différentiel de forme développé dans chacun des formalismes est très fortement similaire. Pourtant, la notion fondatrice de dérivée d'une fonction de forme n'a pas le même sens dans les deux cadres : dans la méthode des vitesses, elle s'apparente à la semidérivée de Hadamard alors que la dérivée de la méthode de perturbation de l'identité est une dérivée de Fréchet usuelle (cf. [20]).

Le troisième point où se rejoignent les deux méthodes et celui de la topologie sur les espaces de formes ou de façon équivalente la caractérisation de la continuité des fonctions du domaine. Cette thématique a déja été abordée dans [20]. Les auteurs y caractérisent la continuité d'une fonction du domaine $J : \Omega \to J(\Omega) \in B$ où B est un espace de Banach et Ω appartient à des espaces de formes Lipschitziennes ou $\mathcal{C}^k$. Ils montrent que la fonction J est continue en un domaine de référence Ω si et seulement si elle est localement bornée et vérifie une condition de continuité directionnelle : si pour tout V appartenant à un espace ad hoc, on a $J(\Omega) = \lim_{t\to 0} J(T_t(V)(\Omega))$.

Dans cette étude, l'admissibilité des domaines ne dépend que de leur régularité ; aucune contrainte géométrique supplémentaire n'est imposée. En conséquence, la détermination de familles à un paramètre joignant deux domaines proches, ce qui est le point essentiel de la démonstration, est fortement simplifiée puisque la famille θ_t, $t \in [0,1]$, définie par l'interpolation linéaire $\theta_t = (1-t)\cdot\theta_0 + t\cdot\theta_1$ va convenir.

Nous reprenons dans ce chapitre une étude analogue en imposant une contrainte supplémentaire : que les domaines Ω admissibles appartiennent à une région de conception D donnée.

Finalement, sous la simple hypothèse que D est un ouvert borné de $\mathbb{R}^n$ de classe $\mathcal{C}^{k+1}$, on établit l'équivalence entre continuité usuelle et continuité directionnelle. On présente également quelques applications typiques de ce résultat.

3 - Régularité générique par rapport à la géométrie des équations quasilinéaires

Ce chapitre constitue la continuation de l'étude du chapitre 1 visant à préciser autant que possible dans quelle mesure l'état des systèmes aux dérivées partielles non-linéaires est différentiable par rapport au domaine. Les équations de Navier-Stokes pour lesquelles l'approche que l'on va développer ne semble pas s'appliquer (ou du moins pas de façon directe) est abandonné au profit d'une équation quasilinéaire scalaire qui présente de nombreuses analogies structurelles.

Comme dans le cas des équations de Navier-Stokes, la dérivée de l'état par rapport à la géométrie existe pour une configuration donnée si les données associées sont suffisamment régulières et si l'équation est non-singulière pour cette configuration, c'est-à-dire si l'équation linéarisée est bien posée. Cette deuxième condition assure également l'unicité locale des solutions de l'équation non-linéaire. C'est pourquoi les méthodes qui permettent de prouver l'existence *générique*, c'est-à-dire sur un ensemble dense de géométries, de la dérivée de l'état par rapport à la forme peuvent être trouvées dans les articles traitant de la finitude générique (par rapport à un paramètre à préciser) du nombre de solutions.

Une première série de résultats de ce type a été établi en se basant sur le théorème de Smale, une version du théorème de Sard en dimension infinie, dédiée aux opérateurs de Fredholm (cf [46] , [42]). Dans [25] sont établis les résultats de finitude générique des solutions de l'équation de Navier-Stokes par rapport au couple (f, g) des forces volumiques et des conditions aux limites ainsi que par rapport à f seul à g fixé. Ils sont complétés dans [26] par un résultat prenant également en compte la variation du paramètre de viscosité ν.

Les résultats que nous établissons dans ce chapitre proviennent de l'utilisation d'un théorème de transversalité (cf. [40], [1]). Cette même méthode a permi d'établir les résultats de finitude générique des solutions de l'équation de Navier-Stokes par rapport aux seules conditions aux limites g (à f et ν fixé) dans [44]. Enfin, dans [43] est prouvée la finitude générique des solutions d'une classe d'équations quasilinéaires par rapport aux paramètres que sont les coefficients de l'équstion, les valeurs aux limites et la géométrie du domaine.

Pour montrer que pour certains jeux de données, l'état u d'une équation quasilinéaire est génériquement (par rapport à la géométrie) dérivable par rapport au domaine, nous prenons comme paramètre une transformation régulière θ d'un domaine de référence Ω sur son image Ω_θ. Toute solution u_θ de l'équation quasilinéaire dans l'ensemble Ω_θ peut alors être caractérisée comme solution d'une équation $\mathcal{A}(u \circ \theta, \theta) = 0$: c'est *l'équation transportée*, définie sur le domaine de référence Ω. Cette formulation rentre dans le cadre de ce théorème. Deux outils sont fondamentaux pour la vérification des hypothèses du théorème du transversalité. Le premier point, qui simplifie notablement la vérification des hypothèses tient aux propriétés de l'équation transportée, plus précisément des liens entre les deux équations transportées associées à deux géométries de référence. Enfin, la conclusion est obtenue par l'utilisation de théorèmes de continuation unique pour les équations quasilinéaires et linéaires (cf. [31], [39], [4]).

4 - Sensibilité de l'équation de Navier-Stokes non-cylindrique par rapport aux perturbations de la géométrie

Historiquement, les premiers problèmes qui firent l'objet d'une analyse de sensibilité par rapport à la géométrie étaient soit intrinsèquement statiques, soit traités uniquement dans le cas stationnaire. On pense ainsi au problème de Didon (ou problème isopérimétrique) ou au problème de Newton, analysé dans l'ouvrage majeur *Philosophiae naturalis principia mathematica* (1687), dont la résolution est la première application de l'optimisation de forme à la mécanique des fluides.

Lors des débuts de la théorie moderne de l'analyse de forme dans les années 70, le cadre théorique ainsi que les méthodes développées sont clairement destinés à l'analyse de système statiques, comme en peut s'en convaincre à la lecture de [50], [47] , [36] , [13], [11], et [12], liste qui est très partielle.

Les études consacrées aux systèmes d'équations aux dérivées partielles ayant une dynamique temporelle se limitent souvent à la recherche d'une géométrie efficace *une fois pour toute* : quelle que soit la configuration géométrique choisie, elle ne changera pas avec le temps et le domaine spatio-temporel associé est alors toujours un cylindre. On pourra consulter [10] pour un exemple d'analyse réalisée dans ce cadre.

En revanche, dans le cas général ou *non-cylindrique*, on considère également les domaines à frontière mobile : il s'agit désormais de contrôle *actif* par rapport à une variable inhabituelle : la géométrie. Les premières études numériques de tels problèmes sont récentes et doivent beaucoup au développement des puissances des ordinateurs. Pourtant, la puissance opératoire nécessaire à de tels traitements est telle que de nombreuses simplifications sont nécessaires pour pouvoir traiter numériquement le modèle (voir par exemple [3] pour la réduction d'ordre de modèle). Les analyses théoriques de tels problèmes sont encore peu nombreuses : l'article [15] est sans doute l'un des premiers s'inscrivant dans cette problématique et les perturbations de la géométrie considérées sont encore très particulières. Plus récemment, les travaux de Raja Dziri et Jean-Paul Zolésio ([22], [23]) ont initié l'analyse de forme non-cylindrique des équations de Navier-Stokes dans un cadre beaucoup plus général. C'est donc à la lumière de ces derniers travaux que je vais situer mes propres contributions.

Pour plus de simplicité dans l'exposition, nous nous restreignons dans cette thèse à l'étude de sensibilité de l'équation de Navier-Stokes par rapport au seul paramètre de la géométrie, et imposons pour ce faire des conditions de Dirichlet homogène sur la frontière du domaine. Les documents cités précédemment intègrent quant à eux la condition au bord physiquement plus réaliste consistant à imposer une vitesse du fluide sur la frontière égale à la vitesse de déplacement des points de cette frontière. Cette formulation résulte en un contrôle hybride par rapport à la géométrie *et au flux imposé sur la frontière*. En effet, deux conditions aux limites distinctes entraînent le même déplacement de la frontière pourvu que leur composantes normales soient identiques.

Dans [22] et [23] la classe des domaines spatio-temporels considérée n'était déterminée qu'indirectement : la variable de contrôle est un champ de vitesse $V(t,x)$ régulier qui détermine l'unique flot $\phi(t,x)$ comme solution de $\partial_t\phi(t,x) = V(t,\phi(t,x))$ et $\phi(0,x) = x$. Les domaines spatio-temporels pouvant être décrits sont de la forme

$$Q_V = \{(t,\phi(t,x)) \in \mathbb{R}\times\mathbb{R}^3,\, t \in]0,T[,\, x\in\Omega\}$$

où Ω est le domaine spatial de départ. Dans un premier temps, nous caractérisons de façon intrinsèque de tels domaines spatio-temporels que nous appelons *ensembles d'évolution*.

Ensuite, pour assurer plus de clarté et l'indépendance du chapitre, à la suite de [23], [41], [8], [38], [27], nous effectuons la démonstration de l'existence d'une solution au problème de Navier-Stokes non-cylindrique.

Par la suite, nous choisissons comme variable de contrôle la transformation θ s'appliquant aux domaines spatio-temporels plutôt que le champ de vitesses V générant ce domaine *à travers le flot*. On simplifie ainsi notablement l'analyse de sensibilité puisque l'on peut accéder directement à la variation du premier ordre $\delta\theta$ de la transformation. L'utilisation de V comme variable de contrôle ne dispense pas du passage par la transformation θ et le calcul de $\delta\theta$ (ou du champ transverse selon la terminologie de [22] et [23] qui sont deux objets étroitement liés) en fonction de δV nécessite alors la résolution d'une EDP.

On établit ensuite un résultat de sensibilité du champ de vitesse u solution de l'équation de Navier-Stokes par rapport à θ sous une hypothèse de régularité des données ainsi que de régularité *de la solution nominale* (c'est-à-dire dans le domaine de référence). Plus précisement, la différentiabilité du champ de vitesse u est établie dans l'espace des fonctions telles que u, ∇u et D^2u appartiennent à $L^2(Q)$ ainsi que dans celui des fonctions telles que $\int_{\Omega_t}|u|^2\,dx$ et $\int_{\Omega_t}|\nabla u|^2\,dx$ soient essentiellement bornées par rapport t. Nous renforçons donc les résultats de régularité obtenus dans [23].

Equations de Navier-Stokes : analyse de forme du cas stationnaire

Ce chapitre est dédié à l'analyse du système stationnaire et incompressible des équations de Navier-Stokes dans un domaine Ω borné de $\mathbb{R}^3$ de frontière Γ

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \quad \text{dans } \Omega \\ \operatorname{div} u = 0 & \quad \text{dans } \Omega \\ u = g & \quad \text{sur } \Gamma \end{aligned}$$

Sensibilité par rapport à la géométrie. L'analyse porte sur la dépendance des solutions du système par rapport à la forme du domaine Ω. Deux classes d'hypothèses sont effectuées :

• des hypothèses de régularité des données (portant sur f, g, la géométrie, la classe de déformations considérée) qui tolèrent en particulier des dépendances complexes des forces et des conditions aux limites en fonction de la géométrie.

• une hypothèse de de non-singularité de l'équation de Navier-Stokes, plus générale que celles de type "haute viscosité" et dont le caractère bien-fondé est discuté. Cette hypothèse ne requiert pas l'unicité des solutions.

Dans ce cadre, on présente deux aspects des résultats de dépendance de la solution par rapport à la géométrie :

• un résultat de perturbation d'une solution donnée dans une géometrie de référence. On insiste en particulier sur l'influence de la régularité des données par rapport aux perturbations de la géométrie sur la régularité des solutions.

• une version multivaluée du théorème précédent où l'on étudie directement l'ensemble des solutions de Navier-Stokes. On met en évidence le nombre fini des solutions et la constance du nombre de ces solutions sous de petites perturbations.

Calculs de Gradients de Forme. On présente deux classes d'applications pour lesquelles les résultats de régularité obtenus sont cruciaux.

• Tout d'abord, les fonctionnelles de domaine *de type Lagrangien*. On étudie le cas où

$$\bar{X}(\Omega) = \frac{1}{T}\int_0^T X(t)\,dt \ \text{ avec } \ X(t) = \int_\Omega \rho(x)\,\phi(t,x)\,dx$$

où ϕ est le flot associé à l'écoulement du fluide:

$$\partial_t \phi(t,x) = u \circ \phi(t,x) \text{ et } \phi(0,x) = x$$

Le vecteur $\bar{X} \in \mathbb{R}^3$ représente la position moyenne d'un polluant convecté par le fluide et dont la densité initiale est $\rho(x)$. La dépendance de cette position en Ω reflète l'influence de la géométrie des parois sur l'écoulement et donc sur l'évolution du polluant. Cette dépendance est explicitée à travers le calcul du gradient de forme de $\bar{X}$.

• On s'intéresse ensuite au calcul du gradient de forme de fonctionnelles ne dépendant que de la contrainte exercée par le fluide sur le bord de son domaine. Il s'agit de fonctionnelles de la forme

$$\Sigma(\Omega) = \int_\Gamma \theta_\Gamma(x, \sigma_n(x))\, d\mathcal{H}^2 \text{ avec } \sigma_n = \langle pI - \nu(Du - Du^*), n\rangle_{\mathbb{R}^3}$$

Le vecteur σ_n est la force surfacique exercée par le fluide sur Γ. On détermine une série d'hypothèses raisonnables portant sur la fonction θ_Γ sous lesquelles le calcul explicite du gradient de forme peut être mené à son terme. On montre en particulier l'application de ces résulats à la fonctionnelle la plus simple de cette famille, à savoir la force totale F exercée sur une portion Λ de Γ.

Chapter 1

The Navier-Stokes Equations: Shape Analysis in the Stationary Case

1.1 The Navier-Stokes Equations: Abstract Setting

We consider the stationary incompressible Navier-Stokes Equations (NSE) in some smooth enough open and bounded sets $\Omega \subset \mathbb{R}^3$ of boundary Γ

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f &\quad \text{in } \Omega \\ \operatorname{div} u = 0 &\quad \text{in } \Omega \\ u = g &\quad \text{on } \Gamma \end{aligned} \tag{1.1}$$

The shape analysis will therefore include as special cases the situation where the flow is uniquely driven by the force field (with homogeneous Dirichlet boundary conditions, cf. [9]), as well as the one where the flow is induced by a body moving at a constant velocity (cf. [6]).

Moreover, for a greater generality, we allow some *a priori* complex dependance from the data (f, g) in the geometry Ω. For a given set Ω, the corresponding value of f, denoted f_Ω, is defined only on Ω, and g_Γ only on Γ.

Abstract Setting. We define the solenoidal space

$$V^1(\Omega) = \{u \in H^1(\Omega;\mathbb{R}^3),\ \operatorname{div} u = 0\} \tag{1.2}$$

where $H^1(\Omega;\mathbb{R}^3)$ is the classical Sobolev space. The trace operator is denoted γ_Γ or simply $(\cdot)|_\Gamma$. Let n be the outer unit normal to Γ. The trace space of $V^{1/2}(\Gamma) = \gamma_\Gamma(V^1(\Omega))$ is given by

$$V^{1/2}(\Gamma) = \left\{g \in H^{1/2}(\Gamma;\mathbb{R}^3),\ \int_\Gamma \langle g, n\rangle \, d\mathcal{H}^2 = 0\right\} \tag{1.3}$$

For convenience, we set $V_0^1(\Omega) = V^1(\Omega) \cap H_0^1(\Omega;\mathbb{R}^3)$ and $V^{-1}(\Omega) = (V_0^1(\Omega))'$.

The linear operator $\pi : H^{-1}(\Omega;\mathbb{R}^3) \to V^{-1}(\Omega)$ is *Leray's projector*: for any $f \in H^{-1}(\Omega;\mathbb{R}^3)$ and any $\phi \in V_0^1(\Omega)$, we have

$$\langle \pi(f), \phi\rangle_{V^{-1}\times V_0^1} = \langle f, \phi\rangle_{H^{-1}\times H_0^1} \tag{1.4}$$

In other words, $\pi(f)$ is the restriction of the linear form f on $H_0^1(\Omega;\mathbb{R}^3)$ to $V_0^1(\Omega)$. For any u and $v \in V^1(\Omega)$, we set

$$Au = -\pi(\Delta u) \;\text{ and }\; B(u,v) = \pi((u\cdot\nabla)v) \tag{1.5}$$

The (nonlinear) Navier-Stokes operator is the mapping

$$\mathfrak{N}: \begin{array}{ccc} V^1(\Omega) & \to & V^{-1}(\Omega)\times V^{1/2}(\Gamma) \\ u & \mapsto & (\nu Au + B(u,u), \gamma_\Gamma(u)) \end{array} \tag{1.6}$$

A solution u of the system (1.1) is *by definition* a function $u \in V^1(\Omega)$ such that $\mathfrak{N}(u) = (\pi(f), g)$. In the sequel, we only consider some *admissible data, i.e.* couples $(f,g) \in H^{-1}(\Omega)\times H^{1/2}(\Gamma)$ such that

$$\int_\Lambda \langle g, n\rangle \, d\mathcal{H}^2 = 0, \;\text{ for any connected component } \Lambda \text{ of } \Gamma \tag{1.7}$$

1.1.1 Shape Analysis framework

Perturbation of Shapes. The shape sensitivity of this equation is studied in the framework of the *Speed Method*. The first step is to generate some of the geometries around the reference set Ω while staying in a given design region. In that purpose, we associate to a smooth, open and bounded set D (designed later on as the *hold-all* or *universe*) a velocity space $\mathcal{V}$, chosen among the $\mathcal{V}_k$ for a $k \geq 1$

$$\mathcal{V}_k = \{V \in \mathcal{C}^0([-T;T];\mathcal{C}^k(\overline{D};\mathbb{R}^3)),\; \langle V, n_D\rangle_{\mathbb{R}^3} = 0 \text{ on } \partial D\} \tag{1.8}$$

Then for any $V \in \mathcal{V}$, a one-parameter family of deformations $T_s : \overline{D} \to \overline{D}$ is given by the following initial-value problem

$$\partial_s T_s = V(s)\circ T_s,\; T_0 = I \tag{1.9}$$

For a given $V \in \mathcal{V}$, we set

$$\Omega_s = T_s(\Omega) \;\text{ and }\; \Gamma_s = T_s(\Gamma) \tag{1.10}$$

They are respectively, the *transported shape* and *transported boundary*. We define the set of admissible shapes by

$$\mathcal{O}_k = \{\Omega \subset D,\, \Omega \text{ open, of class } \mathcal{C}^k\} \tag{1.11}$$

Clearly, for any $V \in \mathcal{V}_k$ and Ω in $\mathcal{O}_k$, Ω_s belongs also to $\mathcal{O}_k$.

Transport of mappings. The regularity of shape-dependent mappings such as f and g are defined in the following way. Let $W(\Omega)$ be either one of the Sobolev spaces $W^{m,p}(\Omega;\mathbb{R}^m)$ ($m \geq 0$, $p \geq 1$) or one of the spaces $\mathcal{C}^k(\overline{\Omega};\mathbb{R}^m)$ ($k \geq 0$).

Definition 1 *We say that the mapping f is $\mathcal{C}^k$ with respect to the shape in $W(\Omega)$ if for any $V \in \mathcal{V}$, the mapping $s \mapsto f_{\Omega_s}\circ T_s$ is in $\mathcal{C}^k(I; W(\Omega))$ on a neighbourhood I of 0.*

The shape sensitivity results of the section 1.2 are expressed in Sobolev spaces whereas the $\mathcal{C}^k$-spaces are needed for the study of the lagrangian functionals (section 1.4).

Of course, an analogous definition holds for the shape-dependent mappings g defined on the boundary Γ. Two different types of derivatives with respect to the geometry may be used to describe the variations of these fields : the *material derivative* of f and g, given by

$$\dot{f}_\Omega = \frac{d}{ds} f_{\Omega_s} \circ T_s|_{s=0} \text{ and } \dot{g}_\Gamma = \frac{d}{ds} g_{\Gamma_s} \circ T_s|_{s=0} \tag{1.12}$$

and the *shape derivative* and *boundary shape derivative*

$$f'_\Omega = \dot{f}_\Omega - Df_\Omega \cdot V(0) \text{ and } g'_\Gamma = \dot{g}_\Gamma - D_\tau g_\Gamma \cdot V(0) \tag{1.13}$$

($D_\tau g$ is the tangential Jacobian matrix of g). When the choice of Ω is clear, the corresponding subscript may be dropped.

Transport by duality. In order to characterize the regularity of shape-dependent forces f that belongs to $H^{-1}(\Omega;\mathbb{R}^3)$, we provide an extension of the previous framework. Let $\phi_s : L^2(\Omega;\mathbb{R}^3) \to L^2(\Omega_s;\mathbb{R}^3)$ be given by

$$\phi_s(f) = f \circ T_s^{-1} \tag{1.14}$$

with $\gamma_s = \det DT_s$ (see also section 1.2.2).

We characterize the regularity of a shape-dependent mapping f with values in $W(\Omega) = H^{-1}(\Omega;\mathbb{R}^3)$ by replacing by $s \mapsto f_{\Omega_s} \circ T_s$ has to be replaced by $s \mapsto \gamma_s^{-1}\phi_s^\star(f_{\Omega_s})$ in the definition 1 as well as in the definition of the material derivative (formula (1.12)).

This extension of the definition is consistent as proved in the lemma 1. The shape derivative of f (when it exists) is still given by the equation (1.13).

Lemma 1 *For any $f \in L^2(\Omega_s;\mathbb{R}^3)$, we have*

$$f \circ T_s = \gamma_s^{-1}\phi_s^\star(f) \tag{1.15}$$

Proof – Indeed for any such f and any $\varpi \in L^2(\Omega;\mathbb{R}^3)$, we have

$$\left\langle \gamma_s^{-1}\phi_s^\star(f), \varpi \right\rangle = \left\langle f, \phi_s^{-1}(\gamma_s^{-1} \cdot \varpi) \right\rangle = \int_{\Omega_s} \langle f \circ T_s, \varpi \rangle_{\mathbb{R}^3} \circ T_s^{-1} \cdot \gamma_s^{-1} \circ T_s^{-1} \, dx$$

As $\gamma_s^{-1} \circ T_s^{-1} = \det(D[T_s^{-1}])$, by a change of variable, we obtain $\left\langle \gamma_s^{-1}\phi_s^\star(f), \varpi \right\rangle = \langle f \circ T_s, \varpi \rangle$. ■

1.2 Sensitivity Results

In this section, we derive a result on the regularity of the solution (u,p) of the Navier-Stokes Equations with respect to a perturbation of the boundary. The first part is dedicated to the description of the set of assumptions and of the statement of the result. The corresponding proof is developed in the sections (1.2.2) to (1.2.4).

1.2.1 Assumptions and Main Statement

Let Ω be our reference set. At first, we describe the regularity needed on the geometrical and functional data to derive our sensitivity result. The degree of regularity is given, in this assumption as well as in the proposition by an integer $k \geq 1$.

ASSUMPTION 1 : REGULARITY OF THE DATA

• Ω and D are $\mathcal{C}^{k+1}$ and the velocity space $\mathcal{V}$ is $\mathcal{V}_{k+2}$. • the mappings f and g are continuous with respect to the shape in $H^{k-1}(\Omega;\mathbb{R}^3)$ and $H^{k+1/2}(\Gamma;\mathbb{R}^3)$ respectively. • the mappings f and g are continuously differentiable with respect to the shape in $H^{k-2}(\Omega;\mathbb{R}^3)$ and $H^{k-1/2}(\Gamma;\mathbb{R}^3)$ respectively.

Remark 1 This set of assumptions is rather designed to handle the high regularity case, even if the methods developed in the following sections could be used as well to study the situations where such a regularity is not available.

The smoothness of the geometry is determined so that the regularity of the solutions of the Navier-Stokes Equations is maximal with respect to the known regularity of f and g.

The assumptions made for these mappings may be surprising at first: the spatial regularity of f and $\dot{f}$ on one hand, of g and $\dot{g}$ on the other hand, are not the same. However this is the usual situation; when f does not depend on Ω, that is when there is a $F \in H^k(D;\mathbb{R}^3)$ such that for any $V \in \mathcal{V}$ $f_{\Omega_s} = F|_{\Omega_s}$, the existence of $\dot{f}$ takes place *a priori* only in H^{k-1}, the material derivative being given by $\dot{f} = \langle Df, V(0)\rangle$. An analogous property holds for g.

An interesting consequence of that gap is that the material and shape derivatives of the data have the same spatial regularity as

$$f'_\Omega = \dot{f}_\Omega - Df_\Omega \cdot V(0) \text{ and } g'_\Gamma = \dot{g}_\Gamma - D_\tau g_\Gamma \cdot V(0)$$

The theorem 1 shows that the solutions u and p of the Navier-Stokes Equations exhibit the same kind of regularity. Therefore, the same property holds for their derivatives with respect to the shape.

The second assumption, less common, results from the non-linearity of the Navier-Stokes Equations: it has no equivalent for the linear shape optimization problems as the linearized problem is automatically well-posed when the initial one is.

ASSUMPTION 2 : REGULARITY OF THE NAVIER-STOKES EQUATIONS

The couple (f_Ω, g_Γ) is a regular value of the Navier-Stokes operator (1.6): for any solution u of the system (1.1), the operator $\mathrm{d}\mathfrak{N}(u)$ is an isomorphism.

Remark 2 Explicitly, the assumption 2 states that for any $k \in H^{-1}(\Omega;\mathbb{R}^3)$, $l \in H^{1/2}(\Gamma;\mathbb{R}^3)$ such that $\int_\Gamma \langle k,n\rangle \, d\mathcal{H}^2 = 0$ and any solution of the Navier-Stokes Equations, the (formal) system

$$\begin{aligned} -\nu\Delta\zeta + (u\cdot\nabla)\zeta + (\zeta\cdot\nabla)u + \nabla\pi &= h \quad \text{in } \Omega \\ \operatorname{div}\zeta &= 0 \quad \text{in } \Omega \\ \zeta &= k \quad \text{on } \Gamma \end{aligned} \tag{1.16}$$

has a unique solution $(\zeta,\pi) \in H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})/\mathbb{R}$.

This assumption is naturally fulfilled in the high viscosity (or small data) case: when ν large enough with respect to the data (f,g) or conversely when f and g are small enough in $H^1(\Omega;\mathbb{R}^3)$ and $H^{1/2}(\Gamma;\mathbb{R}^3)$ for a given viscosity ν, then the linearized Navier-Stokes Equations at the considered solution are well-posed.[1]

In general, without such an assumption on the viscosity, the Navier-Stokes operator is still *generically* non-singular with respect to (f,g) or even with respect to g for a fixed value of f (see [25], [44]). However, the well-posedness of the linearized problem is ensured only in more regular (Hölder or Sobolev) spaces than those considered in the section 1.1.

We may now state the main result of this section:

Theorem 1 *Let the assumptions 1 and 2 be satisfied.*

Let (u,p) be a solution of the NSE system (1.1) in Ω. Then, for any $V \in \mathcal{V}$, there is a neighbourhood I of 0 in $\mathbb{R}$ and a locally unique family $s \in I \mapsto (u_s, p_s)$ of solutions of the Navier-Stokes Equations in Ω_s such that:

(i) Initial Condition: $(u_0,p_0) = (u,p)$

(ii) Regularity: $(\mathcal{U},\mathcal{P}) : [s \mapsto (u_s \circ T_s, p_s \circ T_s)]$ *is such that*
- $\mathcal{U} \in \mathcal{C}^0(I;H^{k+1}(\Omega;\mathbb{R}^3)) \cap \mathcal{C}^1(I;H^k(\Omega;\mathbb{R}^3))$
- $\mathcal{P} \in \mathcal{C}^0(I;H^k(\Omega;\mathbb{R}^3)/\mathbb{R}) \cap \mathcal{C}^1(I;H^{k-1}(\Omega;\mathbb{R}^3)/\mathbb{R})$

(iii) Shape Derivatives: *u' and p' are the solutions of*

$$\begin{aligned} -\nu\Delta u' + (u\cdot\nabla)u' + (u'\cdot\nabla)u + \nabla p' = f' \quad & \text{in } \Omega \\ \operatorname{div} u' = 0 \quad & \text{in } \Omega \\ u' = -\frac{\partial u}{\partial n}\langle V(0),n\rangle + g'_\Gamma \quad & \text{on } \Gamma \end{aligned} \tag{1.19}$$

---

[1] **Technical digression:** the use of a (linear continuous) lifting operator $L : V^{1/2}(\Gamma) \to V^1(\Omega)$ allows to show that $\mathrm{d}F(u)$ is an isomorphism iff for any $f \in V^{-1}(\Omega)$, there is unique solution of

$$\nu A\delta + B(\delta,u) + B(u,\delta) = f, \ \delta \in V_0^1(\Omega) \tag{1.17}$$

This happens in particular when

$$\|u\|_{V^1} \leq \frac{\nu}{2K_cK_s} \tag{1.18}$$

where K_s is the norm of the Stokes operator $A^{-1} : V^{-1}(\Omega) \to V_0^1(\Omega)$ and K_c is such that $\|B(\phi,\psi)\|_{V^{-1}} \leq K_c\|\phi\|_{V^1}\|\psi\|_{V^1}$. Indeed, in that case the mapping $S : \delta \mapsto \nu^{-1}A^{-1}[f - B(u,\delta) - B(\delta,u)]$ is contracting and δ is a fixed point of S iff it is solution of (1.17).

The norm of the solution u in V is itself controlled by the "size" of the data (f,g) (see [48])

Notice that whenever u and $u+\delta$ are solutions of the NSE with the same data (f,g), δ belongs to $V_0^1(\Omega)$ and satisfies $\nu A\delta + B(\delta,u) + B(u,\delta) = 0$. Therefore, the previous threshold also ensures the uniqueness of the solution of the NSE.

It should be pointed out that when the solution of the NSE is not unique, we do not control which one of possible solutions is to be physically selected. For that reason, the study of a the solutions as a set-valued mapping is valuable. We set

$$\mathfrak{u}_s = \{u \in V^1(\Omega_s), u \text{ solution of the NSE in } \Omega_s\} \tag{1.20}$$

and $\mathfrak{u} = \mathfrak{u}_0$.

Lemma 2 *There exist a $\tau > 0$ such that for any $s \in [0, \tau]$, the set $\mathfrak{u}_s$ is finite and*

$$\mathrm{Card}(\mathfrak{u}_s) = \mathrm{Card}(\mathfrak{u}) \tag{1.21}$$

Proof – First we show that every solution u of the Navier-Stokes Equations with data (f, g) in Ω is isolated in $V^1(\Omega)$. Indeed, assume on the contrary that there is a sequence of such solutions u_n such that

$$u_n \to u \text{ in } V^1(\Omega) \text{ and } \forall\, n \in \mathbb{N},\ u_n \neq u \tag{1.22}$$

As $\mathfrak{N}(u) = \mathfrak{N}(u_n) = (f, g)$, the local Taylor expansion at u provides $\mathfrak{N}(u_n) = \mathfrak{N}(u) + \mathrm{d}\mathfrak{N}(u) \cdot (u - u_n) + \varepsilon_n \|u - u_n\|_{V^1}$ with $\varepsilon_n \to 0$ and therefore

$$\|u_n - u\|_{V^1} \leq K\varepsilon_n \|u_n - u\|_{V^1} \text{ with } K = \|[\mathrm{d}\mathfrak{N}(u)]^{-1}\|$$

which yields a contradiction.

Moreover, the set $\mathfrak{u}$ is compact in $V^1(\Omega)$ ([2]). Consequently, $\mathfrak{u}$ is finite. Indeed, as every solution of the NSE is isolated, for any such $u \in \mathfrak{u}$ there is an open set X_u of $V^1(\Omega)$ such that $\mathfrak{u} \cap X_u = \{u\}$. The family $(X_u)_{u \in \mathfrak{u}}$ is a covering of $\mathfrak{u}$ made of open sets. It can therefore be extracted from this family a finite covering, indexed by $\{1, ..., n\}$. The number of solutions in $\mathfrak{u}$ is consequently less or equal to n.

The theorem 1 asserts in particular the existence, for any $u \in \mathfrak{u}$ of an open neighbourhood X_u in $V^1(\Omega)$ and of a $\tau > 0$ such that

$$\forall\, s \in [0, \tau],\ (\mathfrak{u}_s \circ T_s) \cap X_u \text{ has a single element} \tag{1.23}$$

Let $X = \cup_{u \in \mathfrak{u}} X_u$. Assume that there is a decreasing sequence $s_n > 0$, with $s_n \to 0$ when $n \to +\infty$, and a sequence $u_{s_n} \in \mathfrak{u}_{s_n}$ such that

$$\forall\, n \in \mathbb{N},\ u_{s_n} \circ T_{s_n} \in \complement X \tag{1.24}$$

and therefore, $u_{s_n} \circ T_{s_n}$ belongs to a bounded set of $H^2(\Omega)$. Consequently, there is a subsequence (still denoted u_{s_n}) such that $u_{s_n} \circ T_{s_n} \to u$ in $V^1(\Omega)$. hand, (1.24) implies that $u \in \complement X$. This is a contradiction with the very construction of the X and X_u. Therefore, there is no such sequence s_n : there exists a $\tau > 0$ such that for any $s < 0$ and any $u \in \mathfrak{u}_s$, $u \circ T_s$ belongs to one of the X_u. The number of solutions of the NSE is therefore locally constant in s. ■

[2]It is in fact compact in $H^2(\Omega)$ under the assumption 2 : see [44] th. 1.2 p. 158

1.2.2 Transport

To analyse the regularity of the solutions of the NSE with respect to a perturbation of the geometry, we have to compare some mappings whose domain of definition is different. The correspondance between mappings defined on Ω and on Ω_s is obtained by a family of transformations ϕ_s. The choice of

$$\phi_s : u \mapsto u \circ T_s^{-1} \tag{1.25}$$

is adequate for most studies.

The major drawback of these transformations is not to preserve the free-divergence properties of the mappings and the boundary compatibility condition (1.7). As a consequence, the analysis of the transported NSE could not be done *in general* [3] in fixed functional spaces.

In order to circumvent that problem, we introduce the *Piola transformations* in the following definition. Let us set

$$J_s = DT_s \text{ and } \gamma_s = \det J_s \tag{1.26}$$

Definition 2 *Let $V \in \mathcal{V}_1$. For any $s \in [-T;T]$, we define the transformation*

$$\psi_s : L^2(\Omega;\mathbb{R}^3) \to L^2(\Omega_s;\mathbb{R}^3) \tag{1.27}$$

by

$$\psi_s(u) = \phi_s(\gamma_s^{-1} J_s \cdot u) = \frac{D[T_s^{-1}]}{\det D[T_s^{-1}]} \cdot u \circ T_s^{-1} \tag{1.28}$$

The regularity of J_s and γ_s is characterized by the following lemma (see [47]).

Lemma 3 *For any $V \in \mathcal{V}_k$, and any $s \in \mathbb{R}$, J_s and γ_s are invertible. Moreover, J_s and J_s^{-1} are in $\mathcal{C}^1(\mathbb{R};\mathcal{C}^{k-1}(\overline{D};\mathbb{R}^{3\times 3}))$ and γ_s and γ_s^{-1} are in $\mathcal{C}^1(\mathbb{R};\mathcal{C}^{k-1}(\overline{D};\mathbb{R}))$.*

We gather some of the most useful properties shared by ϕ_s and ψ_s in the next proposition.

Proposition 1 *Let $k \in \mathbb{N}$ and $V \in \mathcal{V}_{k+1}$.*

(i) ***Inverse Transformation:*** *ϕ_s and ψ_s are isomorphisms between $L^2(\Omega;\mathbb{R}^3)$ and $L^2(\Omega_s;\mathbb{R}^3)$; their inverse are given by*

$$\phi_s^{-1}(u) = u \circ T_s \text{ and } \psi_s^{-1}(u) = \gamma_s J_s^{-1} \phi_s^{-1}(u) \tag{1.29}$$

(ii) ***Regularity:*** *for any integer $m \leq k$, ϕ_s and ψ_s induce isomorphisms between $H^m(\Omega;\mathbb{R}^3)$ and $H^m(\Omega_s;\mathbb{R}^3)$. As $\mathcal{D}^m(\Omega)$ is also mapped into $\mathcal{D}^m(\Omega_s)$, the adjoint operators $\phi_s^\star$ and $\psi_s^\star$ are isomorphisms from $H^{-m}(\Omega_s;\mathbb{R}^3)$ to $H^{-m}(\Omega;\mathbb{R}^3)$.*

(iii) ***Boundary Regularity:*** *for any integer m such that $m + 1/2 \leq k$, ϕ_s and ψ_s induce isomorphisms between $H^{m+1/2}(\Gamma;\mathbb{R}^3)$ and $H^{m+1/2}(\Gamma_s;\mathbb{R}^3)$.*

Moreover, ψ_s satisfies the two extra following properties:

[3] See however [6] for the case of "wind-tunnel"-like boundary conditions.

Proposition 2 *Let $V \in \mathcal{V}_2$ and $s \in \mathbb{R}$. For any $u \in L^2(\Omega;\mathbb{R}^3)$ and $\varpi \in H_0^1(\Omega;\mathbb{R}^3)$, we have*

$$\langle \operatorname{div} u, \varpi \rangle_{H^{-1}\times H_0^1} = \langle \operatorname{div} \psi_s(u), \phi_s(\varpi) \rangle_{H^{-1}\times H_0^1} \tag{1.30}$$

For any $u \in H^1(\Omega;\mathbb{R}^3)$ and $\varpi \in H^1(\Omega;\mathbb{R}^3)$,

$$\int_{\Gamma_s} \langle \psi_s(u), n_{\Omega_s} \rangle_{\mathbb{R}^3}\, \phi_s(\varpi)\, d\mathcal{H}^2 = \int_\Gamma \langle u, n_\Omega \rangle_{\mathbb{R}^3}\, \varpi\, d\mathcal{H}^2 \tag{1.31}$$

Proof – By a change of variable, we have the equality

$$\begin{aligned} \langle \operatorname{div} \psi_s(u), \phi_s(\varpi) \rangle_{H^{-1}\times H_0^1} &= -\int_{\Omega_s} \langle \psi_s(u), \nabla(\phi_s(\varpi)) \rangle \, dx \\ &= -\int_\Omega \Big\langle \psi_s(u)\circ T_s, \nabla(\phi_s(\varpi))\circ T_s \Big\rangle \gamma_s \, dx \end{aligned}$$

As $\nabla(\phi_s(\varpi))\circ T_s = \nabla(\varpi \circ T_s^{-1})\circ T_s = [DT_s^*]^{-1}\nabla\varpi$, we obtain as desired

$$\begin{aligned} \langle \operatorname{div} \psi_s(u), \phi_s(\varpi) \rangle_{H^{-1}\times H_0^1} &= -\int_\Omega \Big\langle \gamma_s J_s^{-1}\psi_s(u)\circ T_s, \nabla\varpi \Big\rangle \, dx \\ &= -\int_\Omega \langle u, \nabla\varpi \rangle \, dx \\ &= \langle \operatorname{div} u, \varpi \rangle_{H^{-1}\times H_0^1} \end{aligned}$$

To establish the equality 1.31, we make the same change of variable. It yields

$$\int_{\Gamma_s} \langle \psi_s(u), n_{\Omega_s} \rangle\, \phi_s(\varpi)\, d\mathcal{H}^2 \quad = \quad \int_\Gamma \langle \psi_s(u)\circ T_s, n_{\Omega_s}\circ T_s \rangle\, (\phi_s(\varpi)\circ T_s)\, \omega_s\, d\mathcal{H}^2$$

with $\omega_s = \gamma_s \|(J_s^*)^{-1} n_\Omega\|$. As $n_{\Omega_s}\circ T_s = \dfrac{(J_s^*)^{-1} n_\Omega}{\|(J_s^*)^{-1} n_\Omega\|}$, we finally obtain

$$\int_{\Gamma_s} \langle \psi_s(u), n_{\Omega_s} \rangle\, \phi_s(\varpi)\, d\mathcal{H}^2 \quad = \quad \int_\Gamma \langle u, n_\Omega \rangle\, \varpi\, d\mathcal{H}^2$$

■

Corollary 1 *The mapping ψ_s induces an isomorphism between $V^1(\Omega)$ and $V^1(\Omega_s)$, $V_0^1(\Omega)$ and $V_0^1(\Omega_s)$. Consequently, $\psi_s^\star$ induces an isomorphism between $V^{-1}(\Omega_s)$ and $V^{-1}(\Omega)$. Moreover*

$$\psi_s^\star \circ \pi = \pi \circ \psi_s^\star \tag{1.32}$$

Proof – The first statements are consequences of (1.30) and (1.31). Let $\varpi \in V_0^1(\Omega)$ and $f \in H^{-1}(\Omega_s;\mathbb{R}^3)$. We have

$$\langle \psi_s^\star \circ \pi(f), \varpi \rangle_{V^{-1}\times V_0^1} = \langle \pi(f), \psi_s(\varpi) \rangle_{V^{-1}\times V_0^1} = \langle f, \psi_s(\varpi) \rangle_{H^{-1}\times H_0^1}$$

and consequently

$$\langle \psi_s^\star \circ \pi(f), \varpi \rangle_{V^{-1}\times V_0^1} = \langle \psi_s^\star(f), \varpi \rangle_{H^{-1}\times H_0^1} = \langle \pi \circ \psi_s^\star(f), \varpi \rangle_{V^{-1}\times V_0^1}$$

■

Piola derivative. In the process of definition if the material derivative, described in the section 1.1, we could replace the mappings ϕ_s by the ψ_s and therefore defin the *Piola material derivative* of f, $\dot{f}^P$, by

$$\dot{f}_\Omega^P = \frac{d}{ds}\psi_s^{-1}(f_{\Omega_s})|_{s=0} \tag{1.33}$$

In the case of a volume-preserving transformation, that is when $\operatorname{div} V = 0$ and $\gamma_s = 1$, the mapping ψ_s^{-1} reduces to an intrinsic transformation widely used in differential geometry (see for example [7]). The shape derivative of f is then simply deduced from the Piola derivative by the formula

$$f'_\Omega = \dot{f}_\Omega^P - [V(0), f_\Omega] \tag{1.34}$$

where $[\cdot,\cdot]$ are the Lie brackets.

1.2.3 Transported Equation

We prove in this section the part of the theorem 1 dedicated to the existence and uniqueness of $s \mapsto (u_s, p_s)$. We introduce the operators

$$A^s = \psi_s^\star \circ A \circ \psi_s \text{ and } B^s = \psi_s^\star \circ B \circ (\psi_s, \psi_s) \tag{1.35}$$

We call *Transported Navier-Stokes Equations* the system

$$\begin{aligned} \nu A^s u + B^s(u,u) &= \psi_s^\star(\pi(f_{\Omega_s})) \\ u|_\Gamma &= \psi_s^\star(g_{\Gamma_s}) \end{aligned} \tag{1.36}$$

and set

$$\mathfrak{N}^s(u) = (\nu A^s u + B^s(u,u), u|_\Gamma) \tag{1.37}$$

From the very definition of the operators, we have

Proposition 3 *A mapping $u \in V^1(\Omega)$ is a solution of the transported NSE (1.36) if and only if $\psi_s(u)$ is a solution of the NSE with data (f, g) in Ω_s.*

This is the key property for the analysis of $s \mapsto u_s$ which is made first. The regularity with respect of the shape of the pressure is deduced as a consequence in the following section.

1.2.3.1 Regularity of the velocity with respect to the shape

The results concerning $s \mapsto u_s$ are consequences of the implicit function theorem applied to the mapping Ψ defined by

$$\Psi(s,u) = \mathfrak{N}^s(u) - (\psi_s^\star \circ \pi(f_{\Omega_s}), \psi_s^\star(g_{\Gamma_s})) \tag{1.38}$$

The implicit function theorem is in fact applied twice: a first time to obtain the continuity in H^{k+1} and then to get the continuous differentiability in H^k only.

Implicit Function Theorem. Here are the following (classical) versions of the Implicit Function Theorem (I.F.T.) that we use in the sequel.

Let E be a normed space, F and G be some Banach spaces and $f : (x, y) \to f(x, y)$ a continuous application from an open set W of $E \times F$ to G. Finally, let $(a, b) \in W$ such that $f(a, b) = 0$.

Theorem 2 (Basic Assumptions) *Assume that*

(i) *The differential $\partial_y f$ exists on W and is continuous in (a, b).*

(ii) *$\partial_y f(a, b) : F \to G$ is an isomorphism.*

Then, there is an open neighbourhood U of a in E and an open neighbourhood V of b in F such that, for any $x \in U$, the equation $f(x, y) = 0$ has one unique solution in the variable y in V. The mapping $\psi : U \to F$, determined by

$$\forall\, x \in U \quad (\psi(x) \in V) \wedge (f(x, \psi(x)) = 0) \tag{1.39}$$

is continuous.

Theorem 3 (Stronger Assumptions) *Assume that*

(i) *f is continuously differentiable.*

(ii) *$\partial_y f(a, b) : F \to G$ is an isomorphism.*

Then the mapping $\psi : U \to F$ exhibited in the theorem 2 is continuously differentiable.

These statements are (poorly) translated from [RAMIS, t. 3, p. 344-345 and 348]. See [Commande Optimale, Alexéev, Tikhomirov and Fomine, p. 156] for a statement that does not require that $\partial_y f$ is into. See [ZOL, Thèse d'état, p. 134] for a version that involves *weak topologies*.

Application to the shape analysis. In our case, the functional spaces involved in the theorem are

$$\mathcal{F}_k = \left\{u \in H^k(\Omega; \mathbb{R}^3),\ \operatorname{div} u = 0\right\},\ \mathcal{G}_k = \left\{f \in \pi\left(H^{k-2}(\Omega; \mathbb{R}^3)\right)\right\} \tag{1.40}$$

this latter space being a Banach space when endowed by the norm induced by $H^{k-2}(\Omega; \mathbb{R}^3)$([4]), and

$$\mathcal{H}_k = \left\{g \in H^{k-1/2}(\Gamma; \mathbb{R}^3),\ \int_\Gamma \langle g, n\rangle\, d\mathcal{H}^2 = 0\right\} \tag{1.43}$$

Let (u_0, p_0) be the initial solution chosen of the Navier-Stokes Equations in the initial geometry. We have to check the following properties for Ψ:

[4]By definition

$$\|\pi(f)\|_{\mathcal{G}_{k+2}} = \inf_{g \in H^k} \|f + g\|_{H^k} \text{ s.t. } \pi(g) = 0 \tag{1.41}$$

On the other hand, it is classical that

$$(g \in H^k(\Omega; \mathbb{R}^3)\ \wedge\ \pi(g) = 0)\ \Leftrightarrow\ (\exists\, p \in H^{k+1}(\Omega; \mathbb{R}^3) \text{ s.t. } g = \nabla p) \tag{1.42}$$

Thanks to the Hilbert structure of $H^k(\Omega; \mathbb{R}^3)$ and the closedness of $\operatorname{Ker} \pi \cap H^k(\Omega; \mathbb{R}^3)$, it can be shown that the infimum in (1.41) is a minimum: there is a unique $\bar{f}$ of minimal norm s.t. $\pi(\bar{f}) = \pi(f)$. Consequently, $\|\cdot\|_{\mathcal{G}_{k+2}}$ is definite and therefore it is a norm. Moreover for any f, g, we have

$$\|\pi(f) - \pi(g)\|_{\mathcal{G}_{k+2}} = \|\bar{f} - \bar{g}\|_{H^k}$$

It is the key to prove the completedness of $\mathcal{G}_{k+2}$.

(i) The continuity of $\Psi : \mathbb{R} \times \mathcal{F}_{k+1} \to \mathcal{G}_{k+1} \times \mathcal{H}_{k+1}$, the existence of $\partial_u \Psi(s,u)$ anywhere and its continuity in in $(0, u_0)$. The continuous differentiability of Ψ as a mapping from $\mathbb{R} \times \mathcal{F}_k$ to $\mathcal{G}_k \times \mathcal{H}_k$.

(ii) The initial solution u_0 belongs to $\mathcal{F}_{k+1}$ and the mapping $\partial_u \Psi(0, u_0)$ is an isomorphism from $\mathcal{F}_k$ to $\mathcal{G}_k \times \mathcal{H}_k$ as well as from $\mathcal{F}_{k+1}$ to $\mathcal{G}_{k+1} \times \mathcal{H}_{k+1}$.

The properties in (i) are consequences of the explicit expression of the transported operators. Simple calculations show that

$$\psi_s^\star \Delta \psi_s(u) = \gamma_s^{-1} J_s^* \cdot \operatorname{div}(D(\gamma_s^{-1} J_s u)\gamma_s J_s^{-1}(J_s^{-1})^*) \tag{1.44}$$

$$\psi_s^\star((\psi_s(u) \cdot \nabla)\psi_s(v)) = \gamma_s^{-1} J_s^*(u \cdot \nabla)(\gamma_s^{-1} J_s v) \tag{1.45}$$

The desired regularity is obtained from the properties of the trilinear form $(u, v, w) \mapsto \int_\Omega uvw\, dx$ (see [25], [9]). The same properties also yield with a bootstrap method the properties (ii) as the existence is already known for the solution of the Navier-Stokes Equations and also for the linearized equation from the assumption $\mathcal{A}_2$. Notice that the only thing that prevents Ψ to be continuously differentiable from $\mathbb{R} \times \mathcal{F}_{k+1}$ to $\mathcal{G}_{k+1} \times \mathcal{H}_{k+1}$ is the fact that the data f and g are not regular enough.

As a consequence of (i) and (ii), the implicit function theorem asserts that there is a neighbourhood $I \subset \mathbb{R}$ of 0, a unique family $(u_s)_{s \in I}$ of solutions of the Navier-Stokes Equations in Ω_s such that $s \mapsto \psi_s^{-1}(u_s) \in \mathcal{C}^0(I; H^{k+1}(\Omega; \mathbb{R}^3))$ and $s \mapsto \psi_s^{-1}(u_s) \in \mathcal{C}^1(I; H^k(\Omega; \mathbb{R}^3))$. From the definition of ψ_s we deduce that the same regularity is obtained for the mapping $s \mapsto \phi_s^{-1}(u_s)$.

1.2.3.2 Regularity of the pressure

The result of the theorem 1 concerning the pressure is obtained by the same transport methods but applied on the classical form of the Navier-Stokes Equations, without using the projector π. We notice that $\psi_s^\star(\nabla p_s) = \nabla \phi_s^{-1}(p_s)$ and therefore that any solution p_s of the Navier-Stokes Equations in Ω_s is subject to

$$\nabla \phi_s^{-1}(p_s) = \nu \cdot \psi_s^\star \Delta u_s - \psi_s^\star((u_s \cdot \nabla)u_s) + \psi_s^\star(f_{\Omega_s})$$

That equation and the results of the previous section yield the desired regularity of $s \mapsto \phi_s^{-1}(p_s)$.

1.2.4 Shape Derivative and Linearized Equation

In order to establish the equation satisfied by the shape derivatives of u and p, we may use extensions of these shape-dependent mappings. Thanks to the regularity of $s \mapsto u_s$ and $s \mapsto p_s$, there exist two mappings $\bar{u} \in \mathcal{C}^1(\mathbb{R}; H^1(D; \mathbb{R}^3))$ and $\bar{p} \in \mathcal{C}^1(\mathbb{R}; L^2(D; \mathbb{R}))$ such that for s small enough, we have

$$u_{\Omega_s} = \bar{u}(s) \text{ and } p_{\Omega_s} = \bar{p}(s) \text{ on } \Omega_s \tag{1.46}$$

Then, the shape derivatives are given by

$$u'_\Omega = \partial_s \bar{u}(0) \text{ and } p'_\Omega = \partial_s \bar{p}(0) \tag{1.47}$$

However, this method cannot be applied directly to the right-hand side f for the minimal regularity ($k = 1$). But the shape derivative of f may still be characterized weakly by the following lemma.

Lemma 4 *Let f be a shape-dependent mapping which is $\mathcal{C}^0$ w.r. to the shape in $L^2(\Omega;\mathbb{R}^3)$ and $\mathcal{C}^1$ w.r. to the shape in $H^{-1}(\Omega;\mathbb{R}^3)$.*

Let $V \in \mathcal{V}_k$. For any open set ω compactly included in Ω

$$\exists\, \tau \in \mathbb{R}_+^*,\ \forall\, s \in [-\tau;\tau],\ \omega \subset\subset \Omega_s \tag{1.48}$$

Therefore, for any such s, the restriction of f_{Ω_s} to ω does make sense and

$$\frac{\partial}{\partial s} f_{\Omega_s}|_{s=0} = f'_\Omega \ \ in\ \ \mathcal{D}'(\omega)^3 \tag{1.49}$$

Proof – Let $\varpi \in \mathcal{D}(\omega)^3$. As $\phi_s^{-1}(\varpi) = \varpi \circ T_s$, we have

$$\int_\omega \langle f_{\Omega_s}, \varpi\rangle_{\mathbb{R}^3}\ dx = \left\langle \gamma_s^{-1}\cdot \phi_s^\star(f_{\Omega_s}), \gamma_s \cdot \varpi \circ T_s\right\rangle_{H^{-1}\times H_0^1}$$

Both arguments of the duality brackets are strongly differentiable (see "Transport by duality" in the section 1.1.1). That proves the existence of the derivative. By definition of the material derivative in the weak case, we have $\dot f_\Omega = \frac{\partial}{\partial s}\gamma_s^{-1}\cdot \phi_s^\star(f_{\Omega_s})|_{s=0}$. On the other hand

$$\frac{\partial}{\partial s}\gamma_s \cdot \varpi \circ T_s|_{s=0} = \operatorname{div} V(0)\cdot \varpi + D\varpi \cdot V(0) = \operatorname{div}(\varpi \otimes V(0))$$

and therefore, we have

$$\begin{aligned}\frac{\partial}{\partial s}\left(\int_\omega \langle f_{\Omega_s},\varpi\rangle_{\mathbb{R}^3}\ dx\right)\Big|_{s=0} \quad &= \quad \langle \dot f_\Omega, \varpi\rangle_{\mathcal{D}'\times\mathcal{D}} \\ &\quad + \int_\omega \langle f_\Omega, \operatorname{div}(\varpi\otimes V(0))\rangle_{\mathbb{R}^3}\ dx\end{aligned}$$

On the other hand, as

$$\begin{aligned}\int_\omega \langle f_\Omega, \operatorname{div}(\varpi\otimes V(0))\rangle_{\mathbb{R}^3}\ dx \quad &= \quad -\langle Df_\Omega, \varpi \otimes V(0)\rangle_{H^{-1}\times H_0^1} \\ &= \quad -\langle Df_\Omega\cdot V(0), \varpi\rangle_{\mathcal{D}'\times\mathcal{D}}\end{aligned}$$

using the definition $f'_\Omega = \dot f_\Omega - Df_\Omega \cdot V(0)$, we obtain

$$\frac{\partial}{\partial s}\left[\int_\omega \langle f_{\Omega_s},\varpi\rangle_{\mathbb{R}^3}\ dx\right]\Big|_{s=0} = \langle f'_\Omega, \varpi\rangle_{\mathcal{D}'\times\mathcal{D}} \tag{1.50}$$

as needed. ■

Let $\varpi \in \mathcal{D}(\omega)^3$ with ω as in the lemma 4. For $|s| \le \tau$, $\omega \subset\subset \Omega_s$ and therefore

$$\begin{aligned}&\int_\omega \nu\nabla\bar u(s)\cdot\cdot\nabla\varpi + \langle(\bar u(s)\cdot\nabla)\bar u(s), \varpi\rangle_{\mathbb{R}^3}\ dx \\ &\qquad = \int_\omega \bar p(s)\cdot \operatorname{div}\varpi + \langle f_{\Omega_s}, \varpi\rangle_{\mathbb{R}^3}\ dx\end{aligned}$$

The differentiation of this equation with respect to s gives

$$\begin{aligned}&\int_\omega \nu\nabla\partial_s\bar u(0)\cdot\cdot\nabla\varpi + \langle(\partial_s\bar u(0)\cdot\nabla)\bar u(0),\varpi\rangle_{\mathbb{R}^3} + \langle(\bar u(0)\cdot\nabla)\partial_s\bar u(0),\varpi\rangle_{\mathbb{R}^3}\ dx \\ &\qquad = \int_\omega \partial_s\bar p(0)\cdot\operatorname{div}\varpi\ dx + \langle f'_\Omega, \varpi\rangle_{\mathcal{D}'\times\mathcal{D}}\end{aligned}$$

or equivalently, thanks to (1.46) and (1.47)

$$\int_\omega \nu\nabla u' \cdot\cdot \nabla\varpi + \langle (u'\cdot\nabla)u, \varpi\rangle_{\mathbb{R}^3} + \langle u\cdot\nabla)u', \varpi\rangle_{\mathbb{R}^3}\, dx$$
$$= \int_\omega p'\cdot \operatorname{div}\varpi\, dx + \langle f'_\Omega, \varpi\rangle_{\mathcal{D}'\times\mathcal{D}}$$

This is exactly the weak formulation of the first part of the system (1.19).

On the other hand, $\operatorname{div} u_s \equiv 0$ yields that for any $\omega \subset\subset \Omega$, $\varpi \in \mathcal{D}(\omega)$ and any $|s| \le \tau$ (see before), we have

$$\int_\omega \operatorname{div}\bar{u}(s)\cdot\varpi\, dx = 0 \tag{1.51}$$

and it yields, by differentiation with respect to s, that $\operatorname{div} u' = 0$ on Ω in the sense of distributions.

The boundary condition is proved as follows. As $u|_\Gamma = g_\Gamma$, $u'_\Gamma = g'_\Gamma$: indeed, $u'_\Gamma = \dot{u}_\Gamma - D_\tau u\cdot V(0) = \dot{g}_\Gamma - D_\tau g_\Gamma \cdot V(0) = g'_\Gamma$. On the other hand

$$u'_\Gamma = \dot{u}_\Gamma - Du\cdot V(0) + Du(n\otimes n)\cdot V(0) = u'|_\Gamma + \frac{\partial u}{\partial n}\langle V(0), n\rangle$$

therefore, we obtain the desired Dirichlet boundary condition.

1.3 Eulerian Derivatives of Shape Functionals

A major application of the sensitivity results established so far is the calculation of the first-order variation of shape-dependent functionals. Let $k \ge 1$ and $l \ge k$. The set of admissible shapes is $\mathcal{O} = \mathcal{O}_k$ and the velocity space $\mathcal{V} = \mathcal{V}_l$. With these conventions, we make the folowing definition

Definition 3 *A shape functional $J : \mathcal{O} \to \mathbb{R}$ is* shape differentiable *at $\Omega \in \mathcal{O}$ if for any $V \in \mathcal{V}$*

$$dJ(\Omega; V) := \lim_{s\to 0} \frac{J(\Omega_s) - J(\Omega)}{s} \tag{1.52}$$

exists. The value $dJ(\Omega; V)$ is the shape *or* eulerian derivative of J at Ω. *When the choice of V is clear, we also set*

$$J'(\Omega) := dJ(\Omega; V) \tag{1.53}$$

In the cases we will consider, the value of J in Ω is given by

$$J(\Omega) = K(u, p, \Omega) \text{ with } u, p \text{ solution of the NSE in } \Omega \tag{1.54}$$

Strcitly speaking, J is a multi-valued functional. The selection of one branch of solutions (see theorem 1) allows us to consider a single-valued shape functional, at least for small perturbations of the reference set Ω.

1.4 Lagrangian Functionals

We could define the *Lagrangian Functionals* as the class of shape functionals that depends on the trajectories of the fluid particules and not only of the instantaneous values of the fields u, p and their derivatives. A good example of such a functional is $\bar{X}$, given by

$$\bar{X}(\Omega) = \frac{1}{T}\int_0^T X(t)\, dt \text{ avec } X(t) = \int_\Omega \rho(x)\, \phi(t,x)\, dx \tag{1.55}$$

where ϕ is the flow corresponding to the velocity field u, solution of the Navier-Stokes Equations

$$\partial_t \phi(t,x) = u \circ \phi(t,x) \text{ and } \phi(0,x) = x \tag{1.56}$$

The vector $\bar{X}$ is for example the mean position on $[0,T]$ of a pollutant whose initial density is $\rho(x)$.

In order to investigate the sensitivity this shape functional we state some sensitivity results for the solutions of the ordinary differential equations (or ODEs). The analysis is done for time-dependent mappings with values in $\mathcal{C}^k$ spaces rather than in the classical framework of two-variable mappings (time and space).

Before this, we analyze the shape differentiability of a simple distributed shape functional. The corresponding results are needed as tools for the analysis of the lagrangian functionals.

1.4.1 Shape Differentiability of a distributed functional

From now on, we assume that the data considered in the theorem 1 satisfy the assumption 1 (with $k \geq 1$) and 2. Let $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^1$ function whose support is compactly included in the reference set $\Omega_0 \subset D$. We consider the (possibly multi-valued) shape functional J, given by

$$J(\Omega) = \int_\Omega \langle \rho, u \rangle_{\mathbb{R}^3}\, dx \tag{1.57}$$

where u is a solution of the system of equations (1.1). The Reynolds formula (see [47]) yields

$$\mathrm{d}J(\Omega; V) = \int_\Omega \langle \rho, u' \rangle_{\mathbb{R}^3}\, dx \tag{1.58}$$

where u' is the solution of (1.19). The introduction of the corresponding adjoint system allows further simplifications:

$$\begin{array}{rcll} -\nu\Delta\eta - D\eta \cdot u_0 + [Du_0]^* \cdot \eta + \nabla\pi &=& \rho & \text{in } \Omega_0 \\ \operatorname{div} \eta &=& 0 & \text{in } \Omega_0 \\ \eta &=& 0 & \text{on } \Gamma_0 \end{array} \tag{1.59}$$

The assumption 2 ensures the existence and uniqueness of the solution η. The regularity of u, deduced from the assumption 1 and a bootstrapping method shows that $\eta \in H^2(\Omega;\mathbb{R}^3)$. Therefore, we have

$$\begin{aligned}
\mathrm{d}J(\Omega;V) &= \int_\Omega \langle -\nu\Delta\eta - D\eta\cdot u_0 + [Du_0]^*\cdot\eta + \nabla\pi, u'\rangle \ dx \\
&= \int_\Omega \langle \eta, f'_\Omega\rangle \ dx + \int_\Gamma \left\langle -\nu\frac{\partial\eta}{\partial n} + \pi n, u'\right\rangle \ dx \\
&= \int_\Omega \langle \eta, f'_\Omega\rangle \ dx + \int_\Gamma \left\langle -\nu\frac{\partial\eta}{\partial n} + \pi n \ , \ g'_\Gamma - \frac{\partial u}{\partial n}\langle V(0), n\rangle\right\rangle \ dx
\end{aligned}$$

Stress tensor. This expression may be rewritten compactly in terms of *stress tensor*.

Definition 4 *Let $u \in H^1(\Omega;\mathbb{R}^3)$ and $p \in L^2(\Omega;\mathbb{R})$. We define the associated* stress tensor *matrix by*

$$\sigma(u,p) = pI - \nu(Du + Du^*) \tag{1.60}$$

Remember that when u and p are respectively the velocity and pressure fields solution of the NSE in Ω, the force F exerted on the piece Λ of the boundary $\partial\Omega$ is given by

$$F = \int_\Lambda \sigma_n \, d\mathcal{H}^2 \ \text{ with } \ \sigma_n = \langle \sigma(u,p), n\rangle_{\mathbb{R}^3} \tag{1.61}$$

where n is the unit outer normal to Ω.

Some additional information on the structure of the normal component of $\sigma(u,p)$ on the boundary may be obtained. Let D_τ and div_τ be respectively the tangential Jacobian and the tangential divergence operators. We state the

Lemma 5 *The mapping*

$$\frac{\partial^*}{\partial n} : V^1(\Omega)\cap H^2(\Omega)^3 \mapsto L^2(\Gamma;\mathcal{H}^2) \ \ s.t. \ \ \frac{\partial^* u}{\partial n} = Du^*\cdot n \tag{1.62}$$

is a first-order boundary operator and

$$\frac{\partial^* u}{\partial n} = [D_\tau u^* - (\mathrm{div}_\tau u)I]\, n \tag{1.63}$$

Proof – Using the decomposition of the operators D and div on the boundary, we obtain

$$Du = D_\tau u + \frac{\partial u}{\partial n}\otimes n \ \text{ and } \ \mathrm{div}\, u = \mathrm{div}_\tau u + \left\langle \frac{\partial u}{\partial n}, n\right\rangle = 0 \tag{1.64}$$

Consequently, we have

$$\begin{aligned}
[Du^*]n &= \left[D_\tau u^* + n\otimes\frac{\partial u}{\partial n}\right] n \\
&= [D_\tau u^*]n + \left\langle\frac{\partial u}{\partial n}, n\right\rangle n \\
&= [D_\tau u^* - (\mathrm{div}_\tau u)I]\, n
\end{aligned}$$

■

As a direct consequence of that lemma, the derivative of J is equal to

$$\mathrm{d}J(\Omega;V) = \int_\Omega \langle \eta, f'_\Omega\rangle \ dx + \int_\Gamma \left\langle \sigma(\eta,\pi)n \ , \ g'_\Gamma - \frac{\partial u}{\partial n}\langle V(0), n\rangle\right\rangle \ d\mathcal{H}^2 \tag{1.65}$$

1.4.2 Shape derivation of the velocity flow

Now, we investigate the regularity with respect to the shape of a flow based on a shape-dependent mapping u. To do so, we study the (maximal) regularity of an extension Φ of the flow ϕ to $\overline{D}$.

Proposition 4 *Let $\Omega_0 \subset D$ be an open bounded set of class $\mathcal{C}^3$ such that*

$$\overline{\Omega_0} \ \textit{or} \ \complement\overline{\Omega_0} \cap \overline{D} \subset\subset \overline{D} \tag{1.66}$$

Let $V = \mathcal{V}_3$ be the velocity space. Assume that u is continuous with respect to the shape in $\mathcal{C}^3(\overline{\Omega_0};\mathbb{R}^3)$ and continuously differentiable with respect to the shape in $\mathcal{C}^2(\overline{\Omega_0};\mathbb{R}^3)$. Assume also that

$$\textit{for any admissible } \Omega,\ \langle u, n\rangle \equiv 0 \ \textit{ on } \partial\Omega$$

so that the corresponding flow $\phi(\Omega)$ is a mapping from $[-T;T]\times\overline{\Omega}$ to $\overline{\Omega}$. Then, for any $V \in \mathcal{V}$, there is a mapping $\Phi \in \mathcal{C}^1([-T;T];\mathcal{H})$ with $\mathcal{H} = \mathcal{C}^1([-T;T]\times\overline{D};\mathbb{R}^3)$ such that

$$\forall\,(s,t) \in [-T;T]^2,\ \forall\, x \in \overline{\Omega_s},\ \Phi(s)(t,x) = \phi(\Omega_s)(t,x) \tag{1.67}$$

The mapping $\phi' : [-T;T]\times\overline{\Omega_0} \to \overline{\Omega_0}$ such that

$$\forall\, t \in [-T;T],\ \forall\, x \in \overline{\Omega_0},\ \phi'(t,x) := \Phi'(0)(t,x) \tag{1.68}$$

is the solution of the system

$$\partial_t\phi' = u' \circ \phi(\Omega_0) + [\partial_x u \circ \phi(\Omega_0)]\cdot\phi' \ \ \textit{and} \ \ \phi'(0) = 0 \tag{1.69}$$

Proof – The regularity of u yields the existence of a mapping $\bar{u}$ that belongs to $\mathcal{C}^1([-T;T],\mathcal{C}^2(\overline{D};\mathbb{R}^3))$ such that

$$\forall\, s \in [-T;T],\ \forall\, x \in \Omega_s,\ \bar{u}_s(x) = u_{\Omega_s}(x)$$

The shape derivative u'_Ω is equal to $\partial_s\bar{u}(0)|_\Omega$. Thanks to the assumption (1.66), $\bar{u}(s)$ may be taken in $\mathcal{V}_2(D)$. Let $\Phi(s)$ be the flow associated to $\bar{u}(s)$, solution of

$$\begin{gathered}\partial_t\Phi(s) = \bar{u}(s)\circ\Phi(s)\\ \Phi(s) = I\end{gathered} \tag{1.70}$$

Obviously, we have $\Phi(s)(t)|_{\Omega_s} = \phi(\Omega_s)(t)$ and the proposition 53 (appendix A) yields that $\Phi \in \mathcal{C}^1(\mathbb{R};\mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^3)))$ and $\Phi' = \partial_s\Phi(0)$ satisfies

$$\begin{aligned}&\partial_t\Phi' = \partial_s\bar{u}(0)\circ\Phi(0) + [\partial_x\bar{u}(0)\circ\Phi(0)]\cdot\Phi'\\ &\Phi'(0) = 0\end{aligned} \tag{1.71}$$

which yields (1.69). ■

1.4.3 Shape Gradient of a Lagrangian Functional

Let $\hat{\rho} : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^\infty$ function such that $\forall\, t \in [0,T],\ \mathrm{Supp}\,\hat{\rho}(t,\cdot) \subset \Omega$. We consider the shape functional J, defined by

$$J(\Omega) = \iint_{[0,T]\times D} \langle\hat{\rho},\phi\rangle_{\mathbb{R}^3}\, dtdx \tag{1.72}$$

where ϕ is the flow corresponding to the solution u of the Navier-Stokes Equations in Ω. For example, with the choice of

$$\hat{\rho}(t,x) = \frac{1}{T} \cdot \rho(x) e_i \tag{1.73}$$

where e_i is the i-th vector of the canonical basis of $\mathbb{R}^3$, we have clearly

$$J(\Omega) = \bar{X}_i(\Omega) \ \text{ with } \ \bar{X} = (\bar{X}_1, \bar{X}_2, \bar{X}_3) \ \text{ given by (1.55)} \tag{1.74}$$

We assume that the assumptions 1 and 2 are satisfied with $k = 4$. We assume moreover that $\langle g, n_\Omega \rangle_{\mathbb{R}^3} \equiv 0$ and that either $\Omega \subset\subset D$ or $\overline{D} - \overline{\Omega} \subset\subset D$. Under these assumptions, the theorem 1 imply that u satisfies the assumptions of the proposition 4. Consequently, we have the

Proposition 5 *The eulerian derivative of J exists and is given by*

$$\mathrm{d}J(\Omega; V) = \int_\Omega \langle \eta, f'_\Omega \rangle \, dx + \int_\Gamma \left\langle \sigma(\eta,\pi) n \, , \, g'_\Gamma - \frac{\partial u}{\partial n} \langle V(0), n \rangle \right\rangle \, d\mathcal{H}^2 \tag{1.75}$$

where

$$\begin{aligned} &\dot{Q}(t) = -[Du]^* Q(t) - [DQ(t)]u - \hat{\rho}(t,\cdot) \circ \phi(t)^{-1} \\ &Q(T) = 0 \end{aligned} \tag{1.76}$$

and

$$\begin{array}{rcll} -\nu \Delta \eta - D\eta \cdot u + [Du]^* \cdot \eta + \nabla \pi &=& \int_0^T Q(t)\, dt & \textit{in } \Omega \\ \operatorname{div} \eta &=& 0 & \textit{in } \Omega \\ \eta &=& 0 & \textit{on } \Gamma \end{array} \tag{1.77}$$

Proof – We introduce the adjoint equation where the adjoint state is P:

$$\begin{aligned} &\dot{P}(t) = -\hat{\rho}(t,\cdot) - [Du \circ \phi(t)]^* \cdot P(t) \\ &P(T) = 0 \end{aligned} \tag{1.78}$$

Then, the eulerian derivative of J exists and thanks to the equation (1.69) we obtain:

$$\begin{aligned} \mathrm{d}J(\Omega; V) &= \iint_{[0,T]\times\Omega} \langle \hat{\rho}(t,\cdot), \phi'(t) \rangle \, dtdx \\ &= \iint_{[0,T]\times\Omega} \left\langle -\dot{P}(t) - [\partial_x u \circ \phi(t)]^* \cdot P(t) \, , \, \phi'(t) \right\rangle \, dtdx \\ &= \iint_{[0,T]\times\Omega} \langle P(t) \, , \, \partial_t \phi'(t) - [\partial_x u \circ \phi(t)] \phi'(t) \rangle \, dtdx \\ &= \iint_{[0,T]\times\Omega} \langle P(t) \, , \, u' \circ \phi(t) \rangle \, dtdx \end{aligned}$$

This expression may be further simplified by the introduction of $Q(t)$

$$Q(t) = P(t) \circ \phi^{-1} \tag{1.79}$$

which is the solution of the PDE problem (1.76): the differentiation with respect to t of the equation $Q(t)\circ\phi = P(t)$ gives $\dot{Q}(t)\circ\phi(t)+[DQ(t)\circ\phi(t)]\cdot\partial_t\phi(t) = \dot{P}(t)$ and therefore

$$\dot{P}(t)\circ\phi^{-1} = \dot{Q}(t) + DQ(t)\cdot(\partial_t\phi(t)\circ\phi^{-1}(t))$$

The equation (1.76) simply results from the substitution of this equality in (1.78). Then, as $\operatorname{div} u = 0$, $|\det\partial_x\phi(t)t| = 1$ and by a change of variable, we find that

$$\begin{aligned}
\mathrm{d}J(\Omega;V) &= \iint_{[0,T]\times\Omega} \langle P(t)\,,\,u'\circ\phi(t)\rangle\; dtdx \\
&= \iint_{[0,T]\times\Omega} \langle Q(t)\,,\,u'\rangle\; dtdx \\
&= \int_\Omega \left\langle \int_0^T Q(t)\,dt\,,\,u'\right\rangle\, dx
\end{aligned}$$

so as a consequence of the section 1.4.1, we finally obtain the expression of the eulerian derivative given in the proposition 5. ■

1.5 Calculation of the Shape Gradient

1.5.1 Stress-dependent Shape Functionals

In this section, we investigate the shape differentiability of a class of functionals $J(\Omega)$ that depends on the Navier-Stokes flow only through the values of the normal stress on the boundary Γ of Ω. This class includes many useful functionals such as the drag, the lift-to-drag ratio, the moment of the forces exerted on Γ, ... etc.

In the whole section 1.5, we assume that assumptions 1 and 2 of the section 1.2.1 are satisfied with $k = 2$. In the sequel, we simply use the term "admissible" to characterize any set Ω, right-hand side f, etc ... that satisfies these assumptions.

The theorem 1 yields then the following regularity for the reduced[5] stress and normal stress tensor σ and σ_n that are given by

$$\sigma = pI - \nu Du \;\text{ and }\; \sigma_n = pn - \nu\frac{\partial u}{\partial n} \tag{1.80}$$

Lemma 6 *The stress tensor σ belongs to $H^2(\Omega;\mathbb{R}^{3\times3})$ and its material derivative $\dot{\sigma}$ exists in $H^1(\Omega;\mathbb{R}^{3\times3})$. Therefore, the normal stress tensor σ_n belongs to $H^{3/2}(\Gamma;\mathbb{R}^3)$ and its material derivative $\dot{\sigma}_n$ exists in $H^{1/2}(\Gamma;\mathbb{R}^3)$.*

Assume that for any admissible Γ, θ_Γ is a mapping from $\Gamma\times\mathbb{R}^3$ into $\mathbb{R}$. We consider the shape functional J, given by

$$J(\Omega) = \int_\Gamma j_\Gamma\, d\mathcal{H}^2 \;\text{ where }\; j_\Gamma = \theta_\Gamma\circ(I,\sigma_n) \tag{1.81}$$

and where I denotes the identity on Γ.

[5]the missing term $-\nu\frac{\partial^* u}{\partial n}$ in σ_n depends only *and directly* of the boundary value g (see lemma 5). Particularly, if $g = 0$, this extra term vanishes.

Main Example of such a functional. Let (e_i), $i \in \{1, 2, 3\}$ be the canonical basis of $\mathbb{R}^3$ and Λ, the union of some connected components of the boundary Γ. The vector $F = (F_1, F_2, F_3)$ of $\mathbb{R}^3$, with F_i given by

$$F_i(\Omega) = \left\langle \int_\Lambda \sigma_n \, d\mathcal{H}^2 \, , \, e_i \right\rangle_{\mathbb{R}^3} \tag{1.82}$$

is the force exerted by the fluid on Λ. Clearly, we have $J = F_i$ when θ_Γ is given by

$$\theta_\Gamma(x, \sigma) = \chi_\Lambda(x) \cdot \langle \sigma, e_i \rangle \tag{1.83}$$

where χ_A is the characteristic function of the set A.

1.5.2 Regularity of j_Γ. Existence of its material derivative

We introduce the material and shape derivatives of θ_Γ, defined respectively, when the following expressions do make sense, by

$$\dot{\theta}_\Gamma(x, \sigma) := \frac{\partial}{\partial s} \theta_{\Gamma_s}(T_s(x), \sigma) \ , \ \theta'_\Gamma(x, \sigma) = \dot{\theta}_\Gamma(x, \sigma) - \langle \nabla_\tau \theta_\Gamma(x, \sigma), V(0) \rangle \tag{1.84}$$

(∇_τ is the tangential gradient operator on Γ). The family of mappings $s \mapsto T_s$ is the solution of $\partial_s T_s \circ T_s^{-1} = V(s)$ and $T_0 = I$ for a given $V \in \mathcal{V}$; when the dependence of the material and the shape derivatives in the velocity V has to be emphasized, we use the more complete notations $\dot{\theta}_{\Gamma,V}$ and $\theta'_{\Gamma,V}$.

Example of the force exerted on Λ. When θ_Γ is given by the equation (1.83), for any $V \in \mathcal{V}$ and any $s \in [-T; T]$, we have $\chi_{\Lambda_s} \circ T_s = \chi_\Lambda$ with $\Lambda_s = T_s(\Lambda)$. Consequently

$$\dot{\theta}_\Gamma(x, \sigma) = 0 \tag{1.85}$$

As $\nabla_\tau \chi_\Lambda \equiv 0$, we also end up with

$$\theta'_\Gamma(x, \sigma) = 0 \tag{1.86}$$

Roughly speaking, on that example, the mapping θ_Γ does not depend directly on the shape Γ. The value of $j_\Gamma = \theta_\Gamma \circ (I, \sigma_n)$ depends on the shape only through σ_n.

1.5.2.1 Existence and expression of the material derivative of j_Γ.

Proposition 6 *We assume that for any admissible set Ω and corresponding boundary Γ, we have*

- (A1) - *the function θ_Γ is measurable.*
  (A2) - *$\theta_\Gamma(\cdot, 0)$ belongs to $L^1(\Gamma; \mathcal{H}^2)$.*
- (B1) - *θ_Γ is everywhere differentiable in σ.*
  (B2) - *$\partial_\sigma \theta_\Gamma(\cdot, 0)$ belongs to $L^{4/3}(\Gamma; \mathcal{H}^2)$.*
  (B3) - *the family $(\partial_\sigma \theta_\Gamma(x, \cdot))_{x \in \Gamma}$ is equicontinuous.*
- (C1) *for any admissible Γ and $V \in \mathcal{V}$, any $(x, \sigma) \in \Gamma \times \mathbb{R}^3$, the material derivative $\dot{\theta}_{\Gamma,V}(x, \sigma)$ exists and $\dot{\theta}_{\Gamma,V}$ is measurable.*
  (C2) *$\dot{\theta}_{\Gamma,V}(\cdot, 0)$ belongs to $L^1(\Gamma; \mathcal{H}^2)$.*
  (C3) *the mappings $(\sigma, s, W) \to \dot{\theta}_{\Gamma_s,W}(T_s(x), \sigma)$ are equicontinuous in x.*

Then, the mapping $\Gamma \mapsto j_\Gamma$ is shape differentiable in $L^1(\Gamma;\mathcal{H}^2)$ on any admissible set and its material derivative is given by:

$$\frac{\partial}{\partial s}(j_{\Gamma_s} \circ T_s)|_{s=0} = \dot{\theta}_\Gamma \circ (I, \sigma_n) + \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot \dot{\sigma}_n \tag{1.87}$$

Example: the force exerted on Λ. We go back to the example described by the equation (1.83). The properties (A1) and (A2) are obvious. The differentiability of θ_Γ in σ is also clear and

$$\partial_\sigma \theta_\Gamma(x, \sigma) \cdot \delta\sigma = \chi_\Lambda(x) \langle \delta\sigma, e_i \rangle \tag{1.88}$$

That yields easily (B2) and (B3). The properties (C1), (C2) and (C3) are consequences of the equation (1.85). Therefore, we have simply

$$\frac{\partial}{\partial s}(j_{\Gamma_s} \circ T_s)|_{s=0}(x) = \chi_\Lambda(x) \langle \dot{\sigma}_n, e_i \rangle \tag{1.89}$$

Proof of the proposition. The remainder of this section (1.5.2.1) is dedicated to the proof of that proposition. It is constituted of the lemma 7 whose proof is divided in three parts and requires itself two small lemmas, and of the lemma 10.

First, we set

$$R_s = \frac{j_{\Gamma_s} \circ T_s - j_\Gamma}{s} - \dot{\theta}_\Gamma \circ (I, \sigma_n) - \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot \dot{\sigma}_n \tag{1.90}$$

and prove that

Lemma 7 *Let the assumptions of the proposition 6 be satisfied. Then*

$$R_s \to 0 \textit{ in } L^1(\Gamma;\mathcal{H}^2) \textit{ when } s \to 0$$

Proof – We denote $\sigma_n(\Gamma_s)$ the normal stress associated to the solution of the Navier-Stokes Equations in Ω_s and $\sigma_n^s = \sigma_n(\Gamma_s) \circ T_s = \phi_s^{-1}(\sigma_n(\Gamma_s))$.

We split R_s in $R_s = A_s + B_s + C_s$ where

$$A_s = \frac{\theta_\Gamma \circ (I, \sigma_n^s) - \theta_\Gamma \circ (I, \sigma_n)}{s} - \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot \left(\frac{\sigma_n^s - \sigma_n}{s} \right)$$

$$B_s = \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot \left(\frac{\sigma_n^s - \sigma_n}{s} - \dot{\sigma}_n \right)$$

$$C_s = \frac{\theta_{\Gamma_s} \circ (T_s, \sigma_n^s) - \theta_\Gamma \circ (I, \sigma_n^s)}{s} - \dot{\theta}_\Gamma \circ (I, \sigma_n)$$

To prove the lemma, we show that each of these terms tends to 0 in $L^1(\Gamma;\mathcal{H}^2)$ when $s \to 0$

1.5.2.1.1 Limit of $s \mapsto A_s$. As a consequence of the theorem 1 and of the Sobolev injections, we have

$$\sigma_n^s \to \sigma_n \text{ in } L^\infty(\Gamma;\mathcal{H}^2) \text{ and } \frac{\sigma_n^s - \sigma_n}{s} \to \dot{\sigma}_n \text{ in } L^4(\Gamma;\mathcal{H}^2) \tag{1.91}$$

Let us set $\mathcal{A}_s(x) = \{\tau \in \mathbb{R}^3, |\sigma_n(x) - \tau| \le \|\sigma_n - \sigma_n^s\|_{L^\infty}\}$ and

$$\delta_s(x) = \sup_{\tau \in \mathcal{A}_s(x)} |\partial_\sigma \theta_\Gamma(x, \sigma_n(x)) - \partial_\sigma \theta_\Gamma(x, \tau)|$$

Then, we have the inequality

$$|A_s(x)| \le \delta_s(x) \left| \frac{\sigma_n^s(x) - \sigma_n(x)}{s} \right| \tag{1.92}$$

We know that $\delta_s \to 0$ in $L^\infty(\Gamma; \mathcal{H}^2)$ when $s \to 0$: when x describes Γ and $s \in [-\varepsilon; \varepsilon]$, $\sigma_n^s(x)$ takes his values in a bounded subset K of $\mathbb{R}^3$. The assumption (B3) that states the equicontinuity of $(\partial_\sigma \theta_\Gamma(x, \cdot))_{x \in \Gamma}$ yields the *uniform* equicontinuity of $(\partial_\sigma \theta_\Gamma(x, \cdot)|_K)_{x \in \Gamma}$[6] and therefore the desired limit for δ_s.

From the limit given in (1.91), and the equation (1.92), we finally deduce that $\lim_{s \to 0} \|A_s\|_{L^1} = 0$.

1.5.2.1.2 Limit of $s \mapsto B_s$. From the point (B3) and the fact that $\sigma_n \in L^\infty(\Gamma; \mathcal{H}^2)$, we know that $\partial_\sigma \theta_\Gamma \circ (I, \sigma_n) - \partial_\sigma \theta_\Gamma \circ (I, 0) \in L^\infty(\Gamma; \mathcal{H}^2)$. As $\partial_\sigma \theta_\Gamma \circ (I, 0) \in L^{4/3}(\Gamma; \mathcal{H}^2)$ (point (B2)), $\partial_\sigma \theta_\Gamma \circ (I, \sigma_n)$ also belongs to $L^{4/3}(\Gamma; \mathcal{H}^2)$. Therefore, using the Hölder inequality, we get

$$\|B_s\|_{L^1} \le \|\partial_\sigma \theta_\Gamma \circ (I, \sigma_n)\|_{L^{4/3}} \left\| \frac{\sigma_n^s - \sigma_n}{s} - \dot{\sigma}_n \right\|_{L^4} \longrightarrow 0$$

1.5.2.1.3 Limit of $s \mapsto C_s$. For any $x \in \Gamma$, we set $\sigma_x = \sigma_n^s(x)$ and define the function $f_x : [0, s] \to \mathbb{R}$ by

$$f_x(\tau) = \theta_{\Gamma_\tau} \circ (T_\tau(x), \sigma_x) \tag{1.93}$$

where $\tau \mapsto T_\tau$ is the flow associated to the velocity V. For a greater simplicity, we assume that $T = +\infty$ in the definition of the $\mathcal{V}_k$; $V(s)$ is therefore defined for any real value of the variable s and $s \mapsto V(s)$ is uniformly continuous with values in $\mathcal{C}^k(\overline{D}; \mathbb{R}^3)$.

Lemma 9 *Assume that the material derivative of θ_Γ exists everywhere. Then the mapping f_x is everywhere differentiable and its derivative is given by*

$$f_x'(\tau) = \dot{\theta}_{\Gamma_\tau, W(\tau)}(T_\tau(x), \sigma_x) \;\; \textit{with} \;\; W(\tau)(t) = V(t + \tau) \tag{1.94}$$

Proof – This is a consequence of the semi-group properties of the flow. Indeed, if $t \mapsto T_t^\sharp$ denotes the flow associated to the field W defined in the equation (1.94), then we have for any value of τ and t the relation

$$T_{\tau+t} = T_t^\sharp \circ T_\tau \tag{1.95}$$

[6] This is a consequence of the following elementary lemma:

Lemma 8 *Let E be a compact metric space and F a metric space. We consider the family $(f_\lambda)_{\lambda \in \Lambda}$ of mappings from E to F. If it is equicontinuous then it is uniformly equicontinuous.*

Proof – It is a straightforward extension of the classical proof of Heine's theorem. ■

Remark 3 The previous lemma may itself be extended to the following more general result. *Let E be a compact metric space and F, G be metric spaces. If the family $(f_\lambda)_{\lambda \in \Lambda}$ of mappings from $E \times F$ to G is equicontinuous then it is uniformly equicontinuous with respect to its first argument.*

That means simply that for any $y \in F$ and any $\varepsilon > 0$, there is a $\eta > 0$ such that for any $(x, x', y') \in E^2 \times F$ satisfying $d_E(x, x') \le \eta$ and $d_F(y, y') \le \eta$, for any $\lambda \in \Lambda$ we have $d_G(f_\lambda(x, y), f_\lambda(x', y')) \le \varepsilon$.

It is proved as follows. First, we check that for any value of $\tau \in \mathbb{R}$,

$$T_0^\sharp \circ T_\tau = I \circ T_\tau = T_\tau \tag{1.96}$$

Then, we make the calculation $\partial_t(T_t^\sharp \circ T_\tau) = (\partial_t T_t^\sharp) \circ T_\tau = (V(\tau+t) \circ T_t^\sharp) \circ T_\tau = V(\tau+t) \circ (T_t^\sharp \circ T_\tau)$ and we finally end up with

$$\partial_t(T_t^\sharp \circ T_\tau) \circ (T_t^\sharp \circ T_\tau)^{-1} = V(\tau+t) \tag{1.97}$$

The uniqueness of the Cauchy problem determined by the equations (1.96) and (1.97) therefore implies that for any value of τ, the mappings $t \mapsto T_t^\sharp \circ T_\tau$ and $t \mapsto T_{\tau+t}$ are equal.

Now, for a given τ, let us set $\Omega^\sharp = \Omega_\tau$ and $\Gamma^\sharp = \partial\Omega^\sharp = \Gamma_\tau$. We also denote $\Omega_t^\sharp = T_t^\sharp(\Omega^\sharp)$ and $\Gamma_t^\sharp = T_t^\sharp(\Gamma^\sharp)$. Then, we have $\Gamma_{\tau+t} = T_{\tau+t}(\Gamma) = T_t^\sharp \circ T_\tau(\Gamma) = T_t^\sharp(\Gamma_\tau) = T_t^\sharp(\Gamma^\sharp) = \Gamma_t^\sharp$. Consequently,

$$f_x'(\tau) = \frac{\partial}{\partial t}\theta_{\Gamma_{\tau+t}} \circ (T_{\tau+t}(x), \sigma_x)|_{t=0} = \frac{\partial}{\partial t}\theta_{\Gamma_t^\sharp} \circ (T_t^\sharp(T_\tau(x)), \sigma_x)|_{t=0}$$

and therefore $f_x'(\tau) = \dot{\theta}_{\Gamma_\tau, W(\tau)}(T_\tau(x), \sigma_x)$ as desired. ■

From the definition of C_s, we have for any $x \in \Gamma$ the equation

$$\begin{aligned} C_s(x) &= \frac{\theta_{\Gamma_s} \circ (T_s, \sigma_n^s) - \theta_\Gamma \circ (I, \sigma_n^s)}{s} - \dot{\theta}_\Gamma \circ (I, \sigma_n) \\ &= \frac{f_x(0) - f_x(s)}{s} - \dot{\theta}_\Gamma \circ (I, \sigma_n) \end{aligned}$$

and we obtain therefore the inequality

$$|C_s(x)| \leq \sup_{\tau \in [0,s]} |\dot{\theta}_{\Gamma_\tau, W(\tau)}(T_\tau(x), \sigma_n^s(x)) - \dot{\theta}_\Gamma(x, \sigma_n(x))| \tag{1.98}$$

As the field V belongs to $\mathcal{C}^0(\mathbb{R}; \mathcal{C}^k(\overline{D}; \mathbb{R}^3))$, $W(\tau)$ tends to $V = W(0)$ in $\mathcal{V}_k$ when $\tau \to 0$. On the other hand, as $\sigma_n^s \to \sigma_n$ in $L^\infty(\Gamma; \mathbb{R}^2)$, for τ small enough, the set $\{\sigma_n^s(x),\ s \in [0,\tau],\ x \in \Gamma\}$ is included in a compact set K of $\mathbb{R}^3$.

The assumption (C3) states that the mappings $(\sigma, s, W) \to \dot{\theta}_{\Gamma_s, W}(T_s(x), \sigma)$ are equicontinuous in x. As a consequence of the remark 3, they are also uniformly equicontinuous when σ describes the set K and therefore, we conclude that the right-hand side of (1.98) converges to 0 when $s \to 0$ and that $C_s \to 0$ in $L^\infty(\Gamma; \mathcal{H}^2)$ when $s \to 0$. ■

Lemma 10 *Assume that the assumptions of the proposition 6 are satisfied. Then, for a suitable open neighbourhood I of 0 and any $s \in I$*

$$j_{\Gamma_s} \circ T_s \in L^1(\Gamma_s; \mathcal{H}^2) \tag{1.99}$$

Moreover, we also have

$$\dot{\theta}_\Gamma \circ (I, \sigma_n) + \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot \dot{\sigma}_n \in L^1(\Gamma; \mathcal{H}^2) \tag{1.100}$$

Proof – Using the equicontinuity of the $(\partial_\sigma \theta_\Gamma(x, \cdot))_{x \in \Gamma}$ (point (B3)) we obtain the existence of a bound M, uniform in x, such that

$$\text{for any } \sigma \in \mathbb{R}^3 \text{ s.t. } |\sigma| \leq \|\sigma_n\|_\infty,\ |\partial_\sigma \theta_\Gamma(x, \sigma) - \partial_\sigma \theta_\Gamma(x, 0)| \leq M \tag{1.101}$$

Therefore, for any $x \in \Gamma$, we have

$$|j_\Gamma(x)| = |\theta_\Gamma(x, \sigma_n(x))| \leq |\theta_\Gamma(x, 0)| + (|\partial_\sigma \theta_\Gamma(x, 0)| + M) \cdot |\sigma_n(x)|$$

Using the assumptions (A2) ($\theta_\Gamma(\cdot, 0) \in L^1(\Gamma; \mathcal{H}^2)$), (B2) ($\partial_\sigma \theta_\Gamma(\cdot, 0)$ belongs to $L^{4/3}(\Gamma; \mathcal{H}^2)$) and $\sigma_n \in L^\infty(\Gamma; \mathcal{H}^2)$, we obtain that

$$j_\Gamma \in L^1(\Gamma; \mathcal{H}^2) \tag{1.102}$$

Let us prove the property (1.100). The property (1.101) and (B2) provide $\partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \in L^{4/3}(\Gamma; \mathcal{H}^2)$, which, combined with $\dot{\sigma}_n \in L^4(\Gamma; \mathcal{H}^2)$ gives $\partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot \dot{\sigma}_n \in L^1(\Gamma; \mathcal{H}^2)$. On the other hand, using the property (C3), we obtain the uniform equicontinuity of $\sigma \mapsto \dot{\theta}_\Gamma(x, \sigma)$ with respect to x. Combined with the property (C2) ($\dot{\theta}_\Gamma(\cdot, 0) \in L^1(\Gamma; \mathcal{H}^2)$), it gives $\dot{\theta}_\Gamma \circ (I, \sigma_n) \in L^1(\Gamma; \mathcal{H}^2)$. The property (1.100) has been proved.

Let us go back to (1.99). As a consequence of the lemma 7, we have for s small enough a bound of the form $\|R_s\|_{L^1} \leq M$. From the definition of R_s (see (1.90)), the assertions (1.102) and (1.100), we conclude that for such values of s, $j_{\Gamma_s} \circ T_s$ belongs to $L^1(\Gamma_s; \mathcal{H}^2)$ as desired. ■

The proposition 6 is a direct consequence of the lemmas 7 and 10.

1.5.3 Existence and expression of the shape derivative of j_Γ

Corollary 2 *Let the mapping θ_Γ satisfy the assumptions* (C1), (C2) *and* (C3) *of the proposition 6 and moreover assume that for any admissible set Ω of $\mathbb{R}^3$*

$$\bullet \left[\begin{array}{l} \text{(D1) } \theta_\Gamma \in \mathcal{C}^1(\Gamma \times \mathbb{R}^3) \\ \text{(D2) } \nabla_\tau \theta_\Gamma \in L^\infty(\Gamma \times \mathbb{R}^3) \\ \text{(D3) } \partial_\sigma \theta_\Gamma \in L^\infty(\Gamma \times \mathbb{R}^3) \end{array}\right.$$

Then, the shape derivative of j_Γ exists on any such set and is given by

$$j'_\Gamma = \theta'_\Gamma \circ (I, \sigma_n) + \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot (\sigma_n)'_\Gamma \tag{1.103}$$

Example: the force exerted on Λ. The properties (D1) to (D3) are clear in this case. Therefore

$$j'_\Gamma = \chi_\Lambda(x) \, \langle (\sigma_n)'_\Gamma, e_i \rangle \tag{1.104}$$

Remark 4 It is easy to check that the assumptions (D1), (D2) and (D3) imply the (A1), (A2), (B1), (B2), (B3) given in the proposition 6. Therefore, under the assumptions of the corollary 2, the results of this proposition still hold.

The remainder of this subsection is dedicated to the proof of that corollary. The main problem to solve is to obtain the existence in $L^1(\Gamma; \mathbb{R})$ and the expression of $\nabla_\tau j_\Gamma$. To do this, we prove a lemma that extends the chain rule for some Sobolev mappings in open subsets of $\mathbb{R}^n$ (lemma 11). Then, we state the tangential counterpart of this result that is needed (lemma 12) and we conclude the proof.

Lemma 11 *Let U be an open set of $\mathbb{R}^n$. If $F \in \mathcal{C}^1(U \times \mathbb{R}^m; \mathbb{R})$, $\nabla F \in L^\infty(U \times \mathbb{R}^m; \mathbb{R}^{n+m})$ and $G \in W^{1,1}_{\mathrm{loc}}(U; \mathbb{R}^m)$ then $F \circ (I, G) \in W^{1,1}_{\mathrm{loc}}(U; \mathbb{R})$ and its weak gradient is given by the chain rule*

$$\nabla[F \circ (I, G)] = (\nabla_x F) \circ (I, G) + \nabla G \cdot \nabla_y F \circ (I, G) \tag{1.105}$$

Proof – It is an extension of the proof of [24], theorem 4, (ii) p. 129. We check first easily that $F \circ (I, G)$ belongs to $L^1_{\text{loc}}(U; \mathbb{R})$: it is a direct consequence of the inequality

$$|F \circ (I, G) - F \circ (I, 0)| \leq \|\nabla_y F\|_\infty |G|$$

Moreover, the right-hand side of (1.105) belongs clearly to $L^1_{\text{loc}}(U; \mathbb{R})$. Let us prove the equality. Let $V \in \mathcal{C}^1(U; \mathbb{R}^n)$ of compact support. We denote U' a bounded open set such that $\operatorname{Supp} V \subset U' \subset\subset U$. Then, for any small enough $\varepsilon > 0$, we define $G_\varepsilon : U' \to \mathbb{R}^m$ by $G_\varepsilon = \eta_\varepsilon * G$ where η_ε is the standard mollifier[7]. We have $G_\varepsilon \to G$ in $L^1(U'; \mathbb{R}^m)$ and therefore

$$\int_{U'} |[F \circ (I, G) - F \circ (I, G_\varepsilon)] \cdot \operatorname{div} V| \, dx \leq$$
$$\|\operatorname{div} V\|_\infty \cdot \|\nabla_y F\|_\infty \cdot \int_{U'} |G_\varepsilon - G| \, dx \to 0 \text{ when } \varepsilon \to 0$$

Consequently

$$\begin{aligned}
\int_U F \circ (I, G) \cdot \operatorname{div} V \, dx &= \int_{U'} F \circ (I, G) \cdot \operatorname{div} V \, dx \\
&= \lim_{\varepsilon \to 0} \int_{U'} F \circ (I, G_\varepsilon) \cdot \operatorname{div} V \, dx \\
&= -\lim_{\varepsilon \to 0} \left[\int_{U'} \langle (\nabla_x F) \circ (I, G_\varepsilon), V \rangle \, dx \right. \\
&\quad \left. + \int_{U'} \langle \nabla G_\varepsilon \cdot \nabla_y F \circ (I, G_\varepsilon), V \rangle \, dx \right]
\end{aligned}$$

We know that $G_\varepsilon(x) \to G(x)$ $\mathcal{L}^n$ almost everywhere[8]. Moreover, as $|(\nabla_x F) \circ (I, G_\varepsilon)| \leq \|\nabla_x F\|_\infty$, the dominated convergence theorem implies that

$$\int_{U'} \langle (\nabla_x F) \circ (I, G_\varepsilon), V \rangle \, dx \to \int_{U'} \langle (\nabla_x F) \circ (I, G), V \rangle \, dx$$

when $\varepsilon \to 0$. On the other hand,

$$\nabla G_\varepsilon \cdot \nabla_y F \circ (I, G_\varepsilon) = (\nabla G_\varepsilon - \nabla G) \cdot \nabla_y F \circ (I, G_\varepsilon) + \nabla G \cdot \nabla_y F \circ (I, G_\varepsilon)$$

We have the limits

$$\int_{U'} (\nabla G_\varepsilon - \nabla G) \cdot \nabla_y F \circ (I, G_\varepsilon) \, dx \leq \|\nabla_y F\|_\infty \cdot \|\nabla G_\varepsilon - \nabla G\|_{L^1} \to 0$$

(because $G_\varepsilon \to G$ in $W^{1,1}(U'; \mathbb{R}^m)$) and

$$\int_{U'} \nabla G \cdot \nabla_y F \circ (I, G_\varepsilon) \, dx \to \int_{U'} \nabla G \cdot \nabla_y F \circ (I, G) \, dx$$

(again by the dominated convergence theorem: $\nabla G(x) \cdot \nabla_y F(x, G_\varepsilon(x)) \leq |\nabla G(x)| \cdot \|\nabla_y F\|_\infty$). Finally, we have proved the desired equation

$$\int_U F \circ (I, G) \cdot \operatorname{div} V \, dx = -\int_{U'} \langle (\nabla_x F) \circ (I, G) + \nabla G \cdot \nabla_y F \circ (I, G), V \rangle \, dx \tag{1.106}$$

that holds for any compactly supported mapping $V \in \mathcal{C}^1(U; \mathbb{R}^n)$. ■

[7] $\eta_\varepsilon(x) = \dfrac{1}{\varepsilon^n} \eta\left(\dfrac{x}{\varepsilon}\right)$ with $\eta(x) = \begin{cases} c \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$ and c s.t. $\displaystyle\int_{\mathbb{R}^n} \eta(x) \, dx = 1$.

[8] $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$.

Lemma 12 *Let Ω be an open bounded set of $\mathbb{R}^n$ of class $\mathcal{C}^2$. We denote Γ its boundary. If $f \in \mathcal{C}^1(\Gamma \times \mathbb{R}^m;\mathbb{R})$, $\nabla_\tau f \in L^\infty(\Gamma \times \mathbb{R}^m;\mathbb{R})$, $\nabla_y f \in L^\infty(\Gamma \times \mathbb{R}^m;\mathbb{R})$ and $g \in W^{1,1}(\Gamma;\mathbb{R}^m)$ then $f \circ (I, g) \in W^{1,1}(\Gamma;\mathbb{R})$ and its tangential gradient is given by the chain rule*

$$\nabla_\tau[f \circ (I, g)] = (\nabla_\tau f) \circ (I, g) + \nabla_\tau g \cdot \nabla_y f \circ (I, g) \tag{1.107}$$

Proof – Let $h > 0$ be such that ∇b belongs to $\mathcal{C}^2(V;\mathbb{R})$ with

$$U_h = \{x \in \mathbb{R}^n,\, d_\Gamma(x) < h\} \subset\subset V$$

The projector p onto Γ satisfies $p = I - b \cdot \nabla b$ and belongs to $\mathcal{C}^1(\overline{U_h};\mathbb{R})$. We set $U = U_h$, $F(x, y) = f(p(x), y)$ and $G(x) = g(p(x))$. We check that the assumptions of the lemma 11 are satisfied. Clearly, F belongs to $\mathcal{C}^1(U_h;\mathbb{R})$ and we have $\nabla_x F(x, y) = \nabla p(x) \cdot \nabla_\tau f(p(x), y)$ and $\nabla_y F(x, y) = \nabla_y f(p(x), y)$. Therefore ∇F belongs to $L^\infty(U_h \times \mathbb{R}^m;\mathbb{R})$. Finally $G = g \circ p$ belongs to $W^{1,1}(U_h;\mathbb{R}^m)$: this is the intrinsic characterization of the fact that g belongs to $W^{1,1}(\Gamma;\mathbb{R}^m)$ (see appendix B).

As $F \circ (I, G) = f \circ (I, g) \circ p$ belongs to $W^{1,1}_{\text{loc}}(U_h, \mathbb{R})$, $f \circ (I, g)$ belongs to $W^{1,1}(\Gamma;\mathbb{R})$ and its tangential gradient is uniquely characterized by

$$\nabla(F \circ (I, G)) = [I - bD^2 b] \cdot \nabla_\tau[f \circ (I, g)] \circ p$$

(see appendix B). From the lemma 11, we obtain

$$\begin{aligned}
\nabla(F \circ (I, G)) &= (\nabla_x F) \circ (I, G) + \nabla G \cdot \nabla_y F \circ (I, G) \\
&= [I - bD^2 b] \cdot (\nabla_\tau f) \circ (I, g) \circ p \\
&\quad + [I - bD^2 b] \cdot (\nabla_\tau g) \circ p \cdot \nabla_y f \circ (I, g) \circ p \\
&= [I - bD^2 b] \cdot (\nabla_\tau f \circ (I, g) + \nabla_\tau g \cdot \nabla_y f \circ (I, g)) \circ p
\end{aligned}$$

That proves the equation (1.107) and concludes the proof. ■

We finally make the proof of the corollary 2. We apply the lemma 12 with $f = \theta_\Gamma$ and $g = \sigma_n$. The assumptions needed for f are exactly (D1), (D2) and (D3) made in the corollary 2 and $g = \sigma_n$ belongs to $W^{1,1}(\Gamma;\mathbb{R}^3)$ as a consequence of the lemma 6. We obtain therefore that j_Γ belongs to $W^{1,1}(\Gamma;\mathbb{R})$ and the equation

$$\nabla_\tau j_\Gamma = \nabla_\tau \theta_\Gamma \circ (I, \sigma_n) + \nabla_\tau \sigma_n \cdot \nabla_\sigma \theta_\Gamma \circ (I, \sigma_n) \tag{1.108}$$

This, and the proposition 6 that states the existence of the material derivative, yield the existence of the shape derivative. Using the equation (1.108) and the expression (1.87) of the material derivative of j_Γ, we obtain the equality

$$\begin{aligned}
j'_\Gamma &= \frac{\partial}{\partial s}(j_{\Gamma_s} \circ T_s)|_{s=0} - \langle \nabla_\tau j_\Gamma, V(0)\rangle \\
&= \dot{\theta}_\Gamma \circ (I, \sigma_n) + \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot \dot{\sigma}_n \\
&\quad - \langle \nabla_\tau \theta_\Gamma \circ (I, \sigma_n) + \nabla_\tau \sigma_n \cdot \nabla_\sigma \theta_\Gamma \circ (I, \sigma_n), V(0)\rangle \\
&= \dot{\theta}_\Gamma \circ (I, \sigma_n) - \langle \nabla_\tau \theta_\Gamma \circ (I, \sigma_n), V(0)\rangle \\
&\quad + \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot (\dot{\sigma}_n - D_\tau \sigma_n \cdot V(0)) \\
&= \theta'_\Gamma \circ (I, \sigma_n) + \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot (\sigma_n)'_\Gamma
\end{aligned}$$

That concludes the proof.

1.5.4 Eulerian derivative of the shape functional

To begin the explicit computation of the eulerian derivative of the boundary functional J defined by (1.81), we describe in the proposition 7 the general form of the derivatives of such functionals. To do so, we have to show as a prerequisite that the tangential gradient of a mapping $f \in W^{1,1}(\Gamma;\mathbb{R})$ is also the weak gradient of f, defined by transposition over test functions V that are tangent to Γ. This is the purpose of the lemma 53 of the appendix B.

Proposition 7 *Let f_Γ be a such that $f_\Gamma \in W^{1,1}(\Gamma;\mathbb{R})$ and $s \mapsto f_{\Gamma_s} \circ T_s$ is differentiable at $s = 0$ in $L^1(\Gamma;\mathbb{R})$. Let $V_n = \langle V(0), n\rangle$ be the normal component of $V(0)$ and $H = \frac{1}{n-1}\nabla b$ be the (algebraic) mean curvature of Λ.*

The eulerian derivative of the shape functional

$$F(\Omega) = \int_\Gamma f_\Gamma \, d\mathcal{H}^{n-1}$$

exists at Ω and is given by

$$dF(\Omega;V) = \int_\Gamma f'_\Gamma \, d\mathcal{H}^{n-1} + (n-1)\int_\Gamma H f_\Gamma V_n \, d\mathcal{H}^{n-1} \tag{1.109}$$

Proof – By a change of variable, we obtain

$$F(\Omega_s) = \int_{\Gamma_s} f \, d\mathcal{H}^{n-1} = \int_\Gamma f_{\Gamma_s} \circ T_s \cdot \omega(s) \, d\mathcal{H}^{n-1}$$

with $\omega(s) = |M(T_s)\cdot n|$ and where $M(T_s) = \det(DT_s)\,{}^*DT_s^{-1}$ is the cofactor matrix of DT_s. The derivative of $f_{\Gamma_s} \circ T_s$ in $L^1(\Gamma;\mathcal{H}^2)$ is $\dot{f}_\Gamma$ and the derivative of $\omega(s)$ in $\mathcal{C}^1(\Gamma;\mathbb{R})$ is $\mathrm{div}_\tau V(0)$ (see [47], lemma 2.49, p. 80). That proves the existence of $dF(\Omega;V)$ and the formula

$$dF(\Omega;V) = \int_\Gamma \dot{f}_\Gamma \, d\mathcal{H}^{n-1} + \int_\Gamma f_\Gamma \cdot \mathrm{div}_\tau V(0) \, d\mathcal{H}^{n-1} \tag{1.110}$$

Using the equation (B.14) of the lemma 53 and the definition $f'_\Gamma = \dot{f}_\Gamma - \langle \nabla_\tau f, V(0)\rangle$, we obtain the equation (1.109). ■

Proposition 8 *Let $G_\sigma : W^{1,1}(\Gamma;\mathbb{R}) \to L^1(\Gamma;\mathbb{R})$ be the operator defined by*

$$G_\sigma \cdot v := \frac{\partial \sigma}{\partial n} \cdot vn - \sigma \cdot \nabla_\tau v \tag{1.111}$$

Then, if the assumptions of the corollary 2 are satisfied, the eulerian derivative of J is given by

$$\begin{aligned} dJ(\Omega;V) = &\int_\Gamma \theta'_\Gamma \circ (I,\sigma_n) \, d\mathcal{H}^2 + 2\int_\Gamma H \cdot \theta_\Gamma \circ (I,\sigma_n) V_n \, d\mathcal{H}^2 \\ &+ \int_\Gamma \partial_\sigma \theta_\Gamma \circ (I,\sigma_n) \cdot \left(\sigma'_n + G_\sigma \cdot V_n\right) \, d\mathcal{H}^2 \end{aligned} \tag{1.112}$$

To prove that proposition, we first state a version of the Leibniz rule for the computation of the (boundary shape) derivative of a product.

Lemma 13 (Leibniz rule) *Let $p \in [1, +\infty)$ and $p' = p/(p-1)$. Assume that a_Γ and b_Γ be two (boundary) shape-dependent mappings such that*

(i) $a_\Gamma \in W^{1,p}(\Gamma; \mathbb{R})$, $b_\Gamma \in W^{1,p'}(\Gamma; \mathbb{R})$

(ii) $\dot{a}_\Gamma$ *exists in* $L^p(\Gamma; \mathcal{H}^{n-1})$ *and* $\dot{b}_\Gamma$ *in* $L^{p'}(\Gamma; \mathcal{H}^{n-1})$.

Then, the boundary shape derivative of $a_\Gamma b_\Gamma$ exists in $L^1(\Gamma; \mathbb{R})$ and

$$(a_\Gamma b_\Gamma)'_\Gamma = a'_\Gamma b_\Gamma + a_\Gamma b'_\Gamma \tag{1.113}$$

Proof – The bilinear form $(a, b) \in L^p(\Gamma; \mathbb{R}) \times L^{p'}(\Gamma; \mathbb{R}) \mapsto ab \in L^1(\Gamma; \mathbb{R})$ being continuous, $(a_{\Gamma_s} b_{\Gamma_s}) \circ T_s = (a_{\Gamma_s} \circ T_s) \cdot (b_{\Gamma_s} \circ T_s)$ is differentiable in $L^1(\Gamma; \mathbb{R})$ and

$$\partial_s [(a_{\Gamma_s} b_{\Gamma_s}) \circ T_s]|_{s=0} = \dot{a}_\Gamma b_\Gamma + a_\Gamma \dot{b}_\Gamma \tag{1.114}$$

On the other hand, their exists a $h > 0$ such that $a_\Gamma \circ p \in W^{1,p}(U_h; \mathbb{R})$ and $b_\Gamma \circ p \in W^{1,p'}(U_h; \mathbb{R})$ (see appendix B). By Leibniz rule for distributional derivatives, we find $\nabla[(a_\Gamma b_\Gamma) \circ p] = \nabla(a_\Gamma \circ p) \cdot (b_\Gamma \circ p) + (a_\Gamma \circ p) \cdot \nabla(b_\Gamma \circ p)$ and therefore

$$\nabla_\tau (a_\Gamma b_\Gamma) = \nabla_\tau a_\Gamma \cdot b_\Gamma + a_\Gamma \cdot \nabla_\tau b_\Gamma \tag{1.115}$$

The equations (1.114) and (1.115) combined with the definition of the boundary shape derivative yield (1.113). ■

Then, we study the shape derivatives of the unitary outer normal to Ω.

Lemma 14 *Let Ω be a $\mathcal{C}^{1,1}$ open bounded subset of $\mathbb{R}^n$ of boundary Γ. The unitary outer normal n to Ω belongs to $W^{1,\infty}(\Gamma; \mathbb{R}^n)$ and its material derivative exists in $L^\infty(\Gamma; \mathcal{H}^{n-1})$. Its boundary shape derivative also belongs to $L^\infty(\Gamma; \mathcal{H}^{n-1})$ and is given by*

$$n'_\Gamma = -\nabla_\tau V_n \tag{1.116}$$

Proof – The $\mathcal{C}^{1,1}$-regularity of Ω provides the existence of a $h > 0$ such that $\nabla b_\Omega = n \circ p$ belongs to $\mathcal{C}^{0,1}(U_h; \mathbb{R}^n)$ (see appendix B). Therefore, n belongs to $\mathcal{C}^{0,1}(\Gamma; \mathbb{R}^n)$ and consequently to $W^{1,\infty}(\Gamma; \mathbb{R}^n)$ (tangential Rademacher's theorem, see [19], th. 3.2, p. 81). The unitary outer normal n_{Ω_s} to Ω_s is given by

$$n_{\Omega_s} \circ T_s = \frac{{}^*DT_s^{-1} \cdot n_\Omega}{|{}^*DT_s^{-1} \cdot n_\Omega|}$$

and the mapping $\phi : s \mapsto DT_s$ is differentiable in $\mathcal{C}^0(U_h; \mathbb{R}^n)$ with derivative $\partial_s \phi(0) = DV(0)$. Therefore, using the differentiation scheme

$$\frac{\partial}{\partial s} \frac{\lambda_s}{|\lambda_s|} = \frac{1}{|\lambda_s|} \left[\partial_s \lambda_s - \langle \partial_s \lambda_s, \lambda_s \rangle \frac{\lambda_s}{|\lambda_s|} \right]$$

with $\lambda_s = {}^*DT_s^{-1} \cdot n_\Omega$, as $\lambda_0 = n$ and $\partial_s \lambda_s|_{s=0} = -{}^*DV(0) \cdot n$, we end up with

$$\dot{n}_\Gamma = -{}^*DV(0) \cdot n + \langle {}^*DV(0) \cdot n, n \rangle n = -P_\tau({}^*DV(0) \cdot n) \tag{1.117}$$

where $P_\tau = I - n \otimes n$ is the projector onto the tangent space of Γ. On the other hand, $D^2 b_\Omega(x)$ exists for $\mathcal{H}^{n-1}$-almost every $x \in \Gamma$ and $D^2 b_\Omega$ belongs to $L^\infty(\Gamma; \mathcal{H}^{n-1})$ ([19], th. 2.4 p. 66). For any such x

$$\nabla \langle V(0), \nabla b_\Omega \rangle (x) = {}^*DV(0)(x) \cdot \nabla b_\Omega(x) + D^2 b_\Omega(x) \cdot V(0)(x) \tag{1.118}$$

The function $\langle V(0), \nabla b_\Omega \rangle$ being a Lipschitz extension on U_h of $\langle V(0), n \rangle$, we have therefore

$$\nabla_\tau \langle V(0), n \rangle = P_\tau(\nabla \langle V(0), \nabla b_\Omega \rangle) \tag{1.119}$$

(see [19] th. 3.1 p. 77 form (3.11)). Finally, as ∇b_Ω is a Lipschitz extension of n in U_h, we have $D_\tau n = D\nabla b_\Omega \cdot P_\tau = D^2 b_\Omega|_\Gamma = P_\tau \cdot D^2 b_\Omega|_\Gamma$([9]) and therefore

$$-\nabla_\tau \langle V(0), n \rangle = -P_\tau({}^*DV(0) \cdot n) - D_\tau n \cdot V(0)$$

With the definition of the boundary shape derivative and the formula (1.117), it proves the equality (1.116) ■

Finally, we prove the formula the formula (1.112) of the proposition 8. By the corollary 2 and the proposition 7, $dJ(\Omega; V)$ exists. The equations (1.109) and (1.103) yield

$$dJ(\Omega; V) = \int_\Gamma \theta'_\Gamma \circ (I, \sigma_n)\, d\mathcal{H}^2 + 2\int_\Gamma H \cdot \theta_\Gamma \circ (I, \sigma_n) V_n\, d\mathcal{H}^2 + \int_\Gamma \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot (\sigma_n)'_\Gamma\, d\mathcal{H}^2$$

The trace of the stress tensor σ on Γ belongs to $W^{1,2}(\Gamma; \mathbb{R}^{3\times 3})$ (for example) and its material derivative exists in $L^2(\Gamma; \mathcal{H}^2)$. The outer normal n to Ω belongs at least to $W^{1,\infty}(\Gamma; \mathbb{R}^3)$ and its material derivative is defined in $L^\infty(\Gamma; \mathcal{H}^2)$ (lemma 14). Therefore, from the lemma 13 we deduce

$$(\sigma \cdot n)'_\Gamma = \sigma'_\Gamma \cdot n - \sigma \cdot \nabla_\tau V_n = \sigma' \cdot n + \frac{\partial \sigma}{\partial n}(V_n \cdot n) - \sigma \cdot \nabla_\tau V_n$$

This equation, used in the previous expression of $dJ(\Omega; V)$ leads to (1.112).

1.5.4.1 Adjoint system

At that point, we make the following extra assumption on θ_Γ:

- [(E1) there is a $\varepsilon > 0$ such that $\partial_\sigma \theta_\Gamma \in \mathcal{C}^{0,1/2+\varepsilon}_{\mathrm{loc}}(\Gamma \times \mathbb{R}^3)$.

As before, it is clear that the particular case of the force exerted on Λ does satify this assumption.

For a compact $K \subset \mathbb{R}^3$, let us denote by a $\lambda_K \geq 0$ a constant such that for any $(x, y) \in \Gamma^2$ and $(\sigma, \sigma') \in K^2$

$$|\partial_\sigma \theta_\Gamma(x, \sigma) - \partial_\sigma \theta_\Gamma(y, \sigma')| \leq \lambda_K(|x - y|^{\frac{1}{2}+\varepsilon} + |\sigma - \sigma'|^{\frac{1}{2}+\varepsilon}) \tag{1.120}$$

Lemma 15 *Let the assumptions of the corollary 2 and* (E1) *be satisfied. Then the mapping $\partial_\sigma \theta_\Gamma \circ (I, \sigma_n)$ belongs to $H^{1/2}(\Gamma; \mathbb{R}^3)$.*

Proof – In this proof, we use the following characterization of the tangential Sobolev spaces of fractional order s: the mapping $f : \Gamma \to \mathbb{R}^m$ belongs to $W^{s,p}(\Gamma; \mathbb{R}^m)$ for $s \in]0, 1[$ and $p \in (1; +\infty)$ if and only if

$$f \in L^p(\Gamma; \mathcal{H}^2; \mathbb{R}^m) \text{ and } \iint_{\Gamma\times\Gamma} \frac{|f(x) - f(y)|^p}{|x - y|^{2+sp}}\, d\mathcal{H}^2(x) \times d\mathcal{H}^2(y) < +\infty$$

(see for example [2] p. 241, th. 7.48, [30]).

[9]The differentiation of the equality $|\nabla b_\Omega|^2 = 1$ that holds in U_h leads to $D^2 b_\Omega \cdot \nabla b_\Omega = 0$. As $\nabla b_\Omega|_\Gamma = n$, we obtain $D^2 b_\Omega \cdot P_\tau = D^2 b_\Omega|_\Gamma$ and $P_\tau \cdot D^2 b_\Omega|_\Gamma = D^2 b_\Omega|_\Gamma$ by symmetry.

As $\sigma_n \in L^\infty(\Gamma; \mathcal{H}^2)$, there is a compact set K such that $\sigma(x) \in K$ for any $x \in \Gamma$. From the assumption (B3) we know that there is a constant M such that for any $x \in \Gamma$, $|\partial_\sigma \theta_\Gamma(x, \sigma_n(x)) - \partial_\sigma \theta_\Gamma(x, 0)| \leq M$. As $x \mapsto \partial_\sigma \theta_\Gamma(x, 0)$ is Hölder continuous (assumption (E1)), $\mapsto \partial_\sigma \theta_\Gamma(x, \sigma_n(x))$ belongs to $L^2(\Gamma; \mathcal{H}^2)$.

From the inequality

$$|\partial_\sigma \theta_\Gamma(x, \sigma) - \partial_\sigma \theta_\Gamma(y, \sigma')|^2 \leq 2\lambda_K^2 (|x-y|^{1+2\varepsilon} + |\sigma - \sigma'|^{1+2\varepsilon})$$

that is a direct consequence of the relation (1.120), we deduce that

$$\iint_{\Gamma \times \Gamma} \frac{|\partial_\sigma \theta_\Gamma(x, \sigma_n(x)) - \partial_\sigma \theta_\Gamma(y, \sigma_n(y))|^2}{|x-y|^3} \, d\mathcal{H}^2(x) \times d\mathcal{H}^2(y) \leq 2\lambda_K^2 \cdot (A + B) \tag{1.121}$$

with

$$A = \iint_{\Gamma \times \Gamma} \frac{1}{|x-y|^{2-2\varepsilon}} \, d\mathcal{H}^2(x) \times d\mathcal{H}^2(y) \tag{1.122}$$

and

$$B = \iint_{\Gamma \times \Gamma} \frac{|\sigma_n(x) - \sigma_n(y)|^{1+2\varepsilon}}{|x-y|^3} \, d\mathcal{H}^2(x) \times d\mathcal{H}^2(y) \tag{1.123}$$

Clearly, we have $A < +\infty$. On the other hand B is finite if σ_n belongs to $W^{s,p}(\Gamma; \mathbb{R}^3)$ with s and p defined by

$$\begin{cases} 1 + 2\varepsilon = p \\ 3 = 2 + sp \end{cases}$$

As the ε that appears in (1.120) necessarily belongs to $]0, 1/2]$, p belongs to $]1, 2]$ and $s = 1/p$ to $[1/2, 1[$. We know that σ_n belongs to $H^{3/2}(\Gamma; \mathbb{R}^3) = W^{1.5,2}(\Gamma; \mathbb{R}^3)$ and therefore to $W^{s,p}(\Gamma; \mathbb{R}^3)$. Consequently, the right-hand side of (1.121) is finite and the proof is done. ■

Lemma 16 *Let u be a divergence-free function belonging to $H^1(\Omega; \mathbb{R}^3)$. The associated linearized and adjoint Navier-Stokes operators*

$$L \text{ and } L^\sharp : H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}) \to H^{-1}(\Omega; \mathbb{R}^3) \tag{1.124}$$

are defined by the formulas

$$L(\xi, \pi) = -\nu \Delta \xi + Du \cdot \xi + D\xi \cdot u + \nabla \pi \tag{1.125}$$

$$L^\sharp(\zeta, \rho) = -\nu \Delta \zeta + {}^*Du \cdot \zeta - D\zeta \cdot u + \nabla \rho \tag{1.126}$$

We also define the normal stress tensor application

$$\sigma_n : H^2(\Omega; \mathbb{R}^3) \times H^1(\Omega; \mathbb{R}) \to H^{1/2}(\Gamma; \mathbb{R}^3) \tag{1.127}$$

by

$$\sigma_n(\xi, \pi) = \pi n - \nu \frac{\partial \xi}{\partial n} \tag{1.128}$$

Assume that ζ and ξ belong to $H^2(\Omega; \mathbb{R}^3)$, that $\operatorname{div} \xi = \operatorname{div} \zeta = 0$ and that π and ρ belong to $H^1(\Omega; \mathbb{R})$. Then, we have

$$\begin{aligned} \int_\Omega L(\xi, \pi) \cdot \zeta - L^\sharp(\zeta, \rho) \cdot \xi \, dx &= \int_\Gamma \langle \sigma_n(\xi, \pi), \zeta \rangle - \langle \sigma_n(\zeta, \rho), \xi \rangle \, d\mathcal{H}^2 \\ &\quad + \int_\Gamma \langle \xi, \zeta \rangle \langle u, n \rangle \, d\mathcal{H}^2 \end{aligned} \tag{1.129}$$

Proof – The application of Green's formula yields

$$\int_\Omega \langle \Delta\xi, \zeta\rangle \; dx - \int_\Omega \langle \Delta\zeta, \xi\rangle \; dx = \int_\Gamma \left\langle \frac{\partial \xi}{\partial n}, \zeta \right\rangle \; d\mathcal{H}^2 - \int_\Gamma \left\langle \frac{\partial \zeta}{\partial n}, \xi \right\rangle \; d\mathcal{H}^2 \quad (1.130)$$

On the other hand, we easily obtain the equalities

$$\int_\Omega \langle Du\cdot\xi, \zeta\rangle \; dx = \int_\Omega \langle {}^*Du\cdot\zeta, \xi\rangle \; dx \quad (1.131)$$

and

$$\int_\Omega \langle D\xi\cdot u, \zeta\rangle \; dx = -\int_\Omega \langle D\zeta\cdot u, \xi\rangle \; dx + \int_\Gamma \langle \xi, \zeta\rangle \langle u, n\rangle \; d\mathcal{H}^2 \quad (1.132)$$

whereas, using $\operatorname{div}\zeta = \operatorname{div}\xi = 0$, we get

$$\int_\Omega \langle \nabla\pi, \zeta\rangle \; dx = \int_\Gamma \pi \langle \zeta, n\rangle \; d\mathcal{H}^2 \;\; \text{and} \;\; \int_\Omega \langle \nabla\rho, \xi\rangle \; dx = \int_\Gamma \rho \langle \xi, n\rangle \; d\mathcal{H}^2 \quad (1.133)$$

The equations (1.130) to (1.133) finally yield (1.129). ■

We finally add to the previous regularity assumptions the following compatibility condition

- [(F1) For any admissible set Ω, $\int_\Gamma \partial_\sigma\theta_\Gamma \circ (I, \sigma_n)\cdot n \, d\mathcal{H}^2 = 0$

Remark 5 This assumption is natural: as the pressure is defined up to a constant, the value of the shape functional J should not change when σ_n is replaced by $\sigma_n + \lambda n$ and for any value of λ. By differentiation of $J(\Omega)$ with respect to λ, we end up with the property (F1). In the case of the force exterted on Λ, we have

$$\int_\Gamma \partial_\sigma\theta_\Gamma \circ (I, \sigma_n)\cdot n \, d\mathcal{H}^2 = \int_\Gamma \chi_\Lambda \langle n, e_i\rangle \; d\mathcal{H}^2 = \sum_i \int_{\Lambda_i} \langle n, e_i\rangle \; d\mathcal{H}^2$$

where the Λ_i are the connected component of Λ. For any of these Λ_i, let Ω_i be an the open bounded set such that $\partial\Omega_i = \Lambda_i$. We have

$$\int_{\Lambda_i} \langle n, e_i\rangle \; d\mathcal{H}^2 = \int_{\Omega_i} \operatorname{div} e_i \, dx = 0$$

and (F1) is satisfied.

Proposition 9 *Assume that the assumptions of the corollary 2 and* (E1), (F1) *are satisfied. Let* (ζ, ρ) *be the solution of*

$$\begin{aligned} -\nu\Delta\zeta + {}^*Du\cdot\zeta - D\zeta\cdot u + \nabla\rho = 0 &\quad \text{in } \Omega \\ \operatorname{div}\zeta = 0 &\quad \text{in } \Omega \\ \zeta = \partial_\sigma\theta_\Gamma \circ (I, \sigma_n) &\quad \text{on } \Gamma \end{aligned} \quad (1.134)$$

The eulerian derivative of J is given by

$$\begin{aligned} dJ(\Omega; V) = &\int_\Gamma \theta'_\Gamma \circ (I, \sigma_n) \, d\mathcal{H}^2 + 2\int_\Gamma H\cdot\theta_\Gamma \circ (I, \sigma_n) V_n \, d\mathcal{H}^2 \quad (1.135) \\ &+ \int_\Gamma \partial_\sigma\theta_\Gamma \circ (I, \sigma_n)\cdot (G_\sigma\cdot V_n) \, d\mathcal{H}^2 + \int_\Omega f'_\Omega\cdot\zeta \, dx \\ &- \int_\Gamma \left\langle \partial_\sigma\theta_\Gamma \circ (I, \sigma_n) \langle g, n\rangle - \sigma_n(\zeta, \rho), g'_\Gamma - \frac{\partial u}{\partial n} V_n \right\rangle \; d\mathcal{H}^2 \end{aligned}$$

Proof – We already know that u' and p' are solutions of the Navier-Stokes linearized system (cf. theorem 1) with the right-hand side and boundary data

$$L(u', p') = f'_\Omega \text{ and } u' = -\frac{\partial u}{\partial n} \cdot V_n + g'_\Gamma \tag{1.136}$$

On the other hand, the system (1.134) is well-posed: first of all, the Navier-Stokes Equations is assumed to be regular and then, the boundary data is admissible. Indeed, $\partial_\sigma \theta_\Gamma \circ (I, \sigma_n)$ belongs to $H^{1/2}(\Gamma; \mathbb{R}^3)$ (lemma 15) and satisfies the necessary compatibility condition (assumption (F1)). We apply the lemma 16 and obtain

$$\int_\Omega f'_\Omega \cdot \zeta \, dx = \int_\Gamma \langle \sigma_n(u', p'), \zeta \rangle - \langle \sigma_n(\zeta, \rho), u' \rangle \, d\mathcal{H}^2 + \int_\Gamma \langle u', \zeta \rangle \langle u, n \rangle \, d\mathcal{H}^2$$

and therefore

$$\int_\Gamma \langle \sigma'_n, \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \rangle \, d\mathcal{H}^2 = \int_\Omega f'_\Omega \cdot \zeta \, dx$$
$$- \int_\Gamma \left\langle \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \langle g, n \rangle - \sigma_n(\zeta, \rho), g'_\Gamma - \frac{\partial u}{\partial n} V_n \right\rangle d\mathcal{H}^2$$

This equation and the expression (1.112) of the eulerian derivative finally give (1.135). ■

Example of the force exerted on Λ. In that simple case, further simplifications can be made: we have

$$\partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot (G_\sigma \cdot V_n) = \chi_\Lambda \left\langle \frac{\partial \sigma}{\partial n} \cdot V_n n - \sigma \cdot \nabla_\tau V_n, e_i \right\rangle$$

By the tangential Stokes Formula, we have

$$\begin{aligned} -\int_\Lambda \langle \sigma \cdot \nabla_\tau V_n, e_i \rangle \, d\mathcal{H}^2 &= -\int_\Lambda \langle \sigma^\star \cdot e_i, \nabla_\tau V_n \rangle \, d\mathcal{H}^2 \\ &= +\int_\Lambda \operatorname{div}_\tau(\sigma^* \cdot e_i) V_n \, d\mathcal{H}^2 \\ &\quad -\int_\Lambda H \langle \sigma^* \cdot e, n \rangle V_n \, d\mathcal{H}^2 \end{aligned}$$

We may simplify the first expression using

$$\operatorname{div}_\tau(\sigma^* \cdot e_i) = \langle \operatorname{div}_\tau \sigma, e_i \rangle = \left\langle \operatorname{div} \sigma - \frac{\partial \sigma}{\partial n} n \,,\, e_i \right\rangle$$

As on Γ, we have the following decomposition

$$\operatorname{div} \sigma = f - Du \cdot u = f - \left(D_\tau u + \frac{\partial u}{\partial n} n^* \right) \cdot u = f - D_\tau g \cdot g - \frac{\partial u}{\partial n} \langle g, n \rangle$$

we obtain

$$\operatorname{div}_\tau(\sigma^* \cdot e_i) = \left\langle f - D_\tau g \cdot g - \frac{\partial u}{\partial n} \langle g, n \rangle - \frac{\partial \sigma}{\partial n} n \,,\, e_i \right\rangle$$

and therefore

$$\begin{aligned}\int_\Gamma \partial_\sigma \theta_\Gamma \circ (I, \sigma_n) \cdot (G_\sigma \cdot V_n)\, d\mathcal{H}^2 \quad &= \quad \int_\Lambda \left\langle f - D_\tau g \cdot g - \frac{\partial u}{\partial n} \langle g, n \rangle \ , \ e_i \right\rangle V_n \, d\mathcal{H}^2 \\ &\quad - \int_\Lambda H \langle \sigma_n, e_i \rangle V_n \, d\mathcal{H}^2\end{aligned}$$

We finally end up with

$$\begin{aligned}\mathrm{d}J(\Omega; V) \quad = \quad & \int_\Lambda \left\langle f - D_\tau g \cdot g - \frac{\partial u}{\partial n} \langle g, n \rangle \ , \ e_i \right\rangle V_n \, d\mathcal{H}^2 + \int_\Omega f'_\Omega \cdot \zeta \, dx \\ & + \int_\Gamma \left\langle \sigma_n(\zeta, \rho), g'_\Gamma - \frac{\partial u}{\partial n} V_n \right\rangle \, d\mathcal{H}^2 \\ & - \int_\Lambda \left\langle \langle g, n \rangle \, e_i, g'_\Gamma - \frac{\partial u}{\partial n} V_n \right\rangle \, d\mathcal{H}^2\end{aligned} \tag{1.137}$$

Topologies sur les formes. Comparaison des méthodes des vitesses et de perturbation de l'identité

La tâche principale de l'*Analyse de Forme* est l'étude des fonctions J de la forme

$$\Omega \mapsto J(\Omega) : \mathcal{A} \subset \mathfrak{P}(\mathbb{R}^n) \to X \tag{1.138}$$

où $\mathfrak{P}(\mathbb{R}^n) = \{\Omega, \Omega \subset \mathbb{R}^n\}$ et X est un espace topologique: Il s'agit des *fonctions de forme* ou *fonctions de domaine*. Le choix de l'ensemble $\mathcal{A}$ (ensembles mesurables, de périmètre fini, à courbure bornée, ensembles compacts, domaines lisses, etc.) et de la structure associée (simple espace topologique, espace métrique, espace mutationel ou même à l'aide d'une paramétrisation ensemble ouvert d'un espace de Banach,...) est très largement dépendant du problème étudié. On pourra consulter [20] and [5] pour découvrir les structures usuelles.

Continuité dans le cadre des méthodes de transformation. On s'intéresse dans ce chapitre aux concepts de continuité dans le cadre de la *méthode de transformation* ou *de perturbation*. Les deux représentants majeurs de ce cadre sont la *méthode des vitesses* (voir [47]) et la *méthode de perturbation de l'identité* (voir [36]).

Ces deux méthodes ont déja étés partiellement comparées, en particulier le lien entre l'ensemble des transformations généré par ces méthodes ([47],[20] et [18]) ainsi que les notions de *dérivée de forme* que les deux cadres permettent de définir.

Il est crucial de savoir si les définitions de continuité par rapport à la forme qui interviennent naturellement dans ces deux cadres sont équivalentes. Une réponse partielle et affirmative à cette question peut être trouvée dans [20]. La preuve constitue essentiellement en une comparaison de la continuité usuelle et de la continuité le long de courbes, le tout dans des espaces de transformations.

Les choses deviennent toutefois plus délicates lorsque des contraintes additionelles sont imposées à l'ensemble Ω. Dans la plupart des situations, les formes admissibles doivent appartenir à un ensemble D de $\mathbb{R}^n$ appelé *région de conception* ou *univers* qui est un ouvert borné, souvent lisse, de $\mathbb{R}^n$. Le principal résultat de ce chapitre affirme l'équivalence des deux notions de continuité évoquées sous de simples hypothèses de régularité de l'ensemble D.

Applications. A la suite de ce résultat, on propose quelques énoncés permettant de faire le lien entre les formulations de propriétés liées à la continuité dans les deux cadres. En particulier, on caractérise de deux façons :

- les sous-ensembles ouverts des espaces de domaines
- la satisfaction locale d'assertions dépendant de la géométrie
- l'existence d'un minimum local d'une fonctionnelle de domaine

Chapter 2

Comparison of Topologies on Shapes : Velocity and Perturbation of the Identity Methods

2.1 Introduction

The main task of the *Shape Analysis* is the study of the mappings J of the form

$$\Omega \mapsto J(\Omega) : \mathcal{A} \subset \mathfrak{P}(\mathbb{R}^n) \to X \tag{2.1}$$

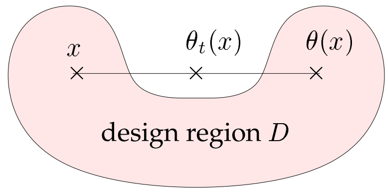
where $\mathfrak{P}(\mathbb{R}^n) = \{\Omega,\, \Omega \subset \mathbb{R}^n\}$ and X is a topological space. These mappings are called *shape functions*. The choice of the set $\mathcal{A}$ (measurable domains, finite perimeter sets, bounded curvature sets, compact sets, smooth domains, etc.) and of its structure (mere topological space, metric space, mutational space or even through a parametrization open subset of a Banach space, ...) is highly dependent of the underlying problem. A rather complete review of the most useful choices is given in [20] and [5].

We concentrate on the continuity issue for the so-called *mapping* or *transformation methods*. Their main representatives are the *velocity* or *speed method* (see [47]) and the *perturbation of the identity* (see [36]). Both methods induce perturbations of a reference set by considering its image under a family of transformations. These transformation methods have proved their efficiency in the analysis of many shape optimization problems (see for example [9],[29],[10]). Their framework is generic: various regularity and constraints of the shapes may be considered by the choice of the family of transformations.

Some parts of these two frameworks have already been compared. In [47], the authors characterize, for a given picewise smooth open and bounded design region $D \subset \mathbb{R}^n$, the exact regularity of a one-parameter family T_t, $t \in [0, \varepsilon]$, of $\mathcal{C}^k$-diffeomorphisms of $\overline{D}$ that is required to obtain this family as the flow of a velocity field $V \in \mathcal{C}^0([0,\varepsilon]; \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$ such that $\langle V(t,x), n(x)\rangle_{\mathbb{R}^n} = 0$ at any point $x \in \partial D$ where the normal is defined. Roughly speaking, the two parameters that are the velocity fields and the transformations of the design

Example 1: It is clear that for the bean-shaped domain pictured on the right, we can find a transformation θ of $\overline{D}$ arbitrarily close to the identity that *does not* satisfy the following stability requirement

$$\theta_t(D) \subset D \text{ for any } t \in [0,1] \tag{2.3}$$



region are equivalent as they may be deduced one from the other. This result is generalized in [20] and [18] to the Lipschitz transformations and velocity fields and to arbitrary open design regions of $\mathbb{R}^n$.

The notions of *shape derivatives* that are defined in both frameworks are similar but not equivalent. The derivative with respect to a velocity field is the generalisation of the Hadamard semiderivative whereas the derivative with respect to a perturbation is a usual Fréchet derivative (see [20]).

In the same spirit, we compare the continuity of shape functions in these two frameworks. Are the continuity with respect to the perturbation and the directional continuity along the paths generated by velocity fields equivalent ? A partial and affirmative answer has been given in [20] in the unconstrained case. The authors consider the shape fonctions $J : \Omega \to B$, where B is a Banach space and Ω is either a $\mathcal{C}^k$ or a Lipschitz open set of $\mathbb{R}^n$. They prove that such a shape function that is locally bounded around a reference set Ω is continous with respect to the perturbation if and only if for any admissible velocity field and corresponding flow T_t, we have $J(\Omega) = \lim_{t\to 0} J(T_t(\Omega))$.

A major tool in this analysis is the abiliy to associate to an admissible transformation θ sufficiently near from the identity a regular path of admissible transformations that joins θ and I. In the previous situation, it is clear that the simple linear interpolation works : the family

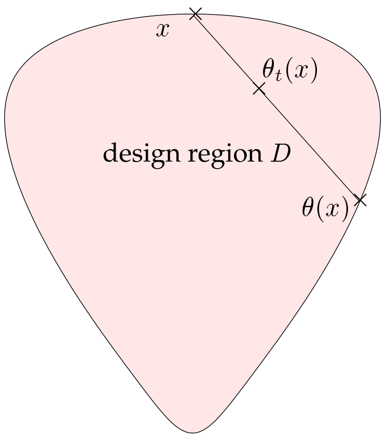
$$\theta_t = (1-t)I + t\theta,\ t \in [0,1] \tag{2.2}$$

is adequate.

The things get more involved when additional constraints are imposed on the set Ω. In most practical situations, the admissible shapes must belong to a set D of $\mathbb{R}^n$, called *hold-all*, *design region* or *universe*, which is often an open, bounded and possibly smooth set of $\mathbb{R}^n$. The admissible transformations are then $\mathcal{C}^k$-diffeomorphisms of $\overline{D}$ and it becomes clear that the linear interpolation given by (2.2) is no longer adequate in general (see the examples 1 and 2).

To overcome this problem, we introduce a pointwise nonlinear interpolation mapping that is used to define a functional interpolation between close transformations of $\overline{D}$. The main result of this document states the equivalence of these two continuity notions for $\mathcal{C}^k$ transformations and velocity fields under the simple assumption that the set D is a $\mathcal{C}^{k+1}$ open bounded set of $\mathbb{R}^n$. Some applications of this result are provided as well.

Structure of the document. We give a brief account of both frameworks in the section 2.2. The section 2.3 is dedicated to the construction of a pointwise interpolation mapping on the hold-all. The specific axioms that we require for that mapping allow to build a suitable functional interpolation mapping in the



Example 2: An additional convexity assumption on D does ensure the stability of this design region characterized by the equation (2.3). However if the boundary of D is everywhere the graph of a strictly convex function, the stability requirement for ∂D, given by

$$\partial \theta_t(D) \subset \partial D \text{ for any } t \in [0,1] \quad (2.4)$$

cannot be met for θ_t unless the transformation θ is the identity on ∂D.

section 2.4. It is the main tool that is needed to compare the two continuity concepts for the shape functions in the section 2.5.4. We conclude by the presentation of some reformulations and applications of that result.

2.2 The Transformation Method Framework and Continuity

As pointed out in the introduction, the transformation methods are generic. For the sake of simplicity, we only consider here the case of $\mathcal{C}^k$ mappings. For any open set $A \subset \mathbb{R}^n$ and Banach space E, $\mathcal{C}^k(\overline{A}; E)$ is the set of functions from A to E whose derivatives of order up to k are uniformly continuous and bounded. The usual norm $\|\cdot\|_k$ gives to $\mathcal{C}^k(\overline{A}; E)$ a Banach space structure.

The corresponding admissible sets belong to

$$\mathcal{A} = \{\Omega \subset D,\, \Omega \text{ open, bounded, of class } \mathcal{C}^k\} \quad (2.5)$$

for a given where $k \in \mathbb{N}^*$ and open subset D of class $\mathcal{C}^k$ of $\mathbb{R}^n$.

Velocity Method. The velocity method builds one-parameter families of transformations $T_s(V) : \overline{D} \to \overline{D}$, generated by a velocity field V which belongs to the *Velocity Space*

$$\mathcal{V}_k(D) = \{V \in \mathcal{C}^0([0,1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^n)),\ \langle V, n\rangle = 0 \text{ on } \partial D\} \quad (2.6)$$

where $n : \partial D \to \mathbb{R}^n$ is the unit outer normal to D. The mapping $s \mapsto T_s(V)$ is defined as the flow associated to V

$$T_s(V)(x) = \xi(s) \text{ with } \frac{d\xi}{ds}(s) = V_s(\xi(s)) \text{ and } \xi(0) = x \quad (2.7)$$

and we associate to a reference set $\Omega \in \mathcal{A}$ the family $\Omega_s(V)$, $s \in [0,1]$, defined by

$$\Omega_s(V) = T_s(V)(\Omega) \quad (2.8)$$

This naturally induces the first definition of continuity for a shape functional $J : \mathcal{A} \to \mathbb{R}$.

Definition 1 *The shape function* $J : \mathcal{A} \to X$ *is* directionally continuous *at* Ω *if for any* $V \in \mathcal{V}_k(D)$

$$J(\Omega) = \lim_{s \to 0} J(\Omega_s(V)) \tag{2.9}$$

Perturbation of the identity. In this framework, the set of transformations is given by

$$\mathrm{Diff}_k(\overline{D}) = \{\theta \in \mathcal{C}^k(\overline{D}; \mathbb{R}^n),\ \theta^{-1} \in \mathcal{C}^k(\overline{D}; \mathbb{R}^n)\} \tag{2.10}$$

Every element θ of this set is a $\mathcal{C}^k$-diffeomorphism from $\overline{D}$ into itself. This set is endowed with the usual $\mathcal{C}^k$ topology. For any $\theta \in \mathrm{Diff}_k(\overline{D})$ and $r > 0$, we set

$$B_k(\theta, r) = \{\theta' \in \mathrm{Diff}_k(\overline{D}),\ \|\theta' - \theta\|_k < r\} \tag{2.11}$$

Let Ω_θ be the image of Ω by the transformation θ and $\mathbf{\Omega} : \mathrm{Diff}_k(\overline{D}) \to X$ the mapping defined by $\mathbf{\Omega}(\theta) = \Omega_\theta$. This mapping induces a topology on $\mathcal{A}$ (see the appendix 2.6). For that topology, a basis of neighbourhoods of the shape Ω is the family $(V_r)_{r>0}$ with

$$V_r = \mathbf{\Omega}(B_k(I, r)) \tag{2.12}$$

The associated continuity is characterized by the following definition

Definition 2 *The shape function* $J : \mathcal{A} \to X$ *is* continuous *at* Ω *if and only if the mapping*

$$\theta \to J(\Omega_\theta) : \mathrm{Diff}_k(\overline{D}) \to X \tag{2.13}$$

is continuous at $\theta = I$.

2.3 Interpolation Mapping

2.3.1 Axioms and Statement of the Result

Let A be an open subset of $\mathbb{R}^n$. We say that ζ *is a k-regular interpolation mapping on* $\overline{A}$ if

(A) there is an open subset X of $A \times A$ such that

$$\mathrm{diag}(A \times A) = \{(x, x),\ x \in A\} \subset X$$

(B) for any $s \in [0, 1]$, ζ_s is a mapping from $\overline{X}$ to $\mathbb{R}^n$.

(C) the following axioms are satisfied:

(i) Interpolation: *for any* s *in* $[0, 1]$ *and any* (x, y) *in* $\overline{X}$,

$$\zeta_0(x, y) = x,\ \zeta_1(x, y) = y \text{ and } \zeta_s(x, x) = x$$

(ii) Stability: *for any* $s \in [0, 1]$, ζ_s *maps* $\overline{X}$ *in* $\overline{A}$ *and* $\overline{X} \cap (\partial A)^2$ *in* ∂A.

(iii) Regularity: $s \mapsto \zeta_s \in \mathcal{C}^1([0; 1]; \mathcal{C}^k(\overline{X}; \mathbb{R}^n))$.

The expression $\zeta_s(x, y) = (1 - s)x + sy$ satisfies these axioms when $A = \mathbb{R}^n$ and $X = A \times A$. However, for most sets A, the stability requirements cannot be met for the linear interpolation, and by any choice of the set X.

Nevertheless, we give a proof of the existence of an interpolation mapping for a large class of geometries of A and an *ad hoc* choice of X. Notice that in general, one cannot obtain the existence of a global interpolation, *i.e.* when $X = A \times A$. For instance if A is a non-simply connected subset of $\mathbb{R}^n$, the axioms

(i) to (iii) cannot be satisfied with $X = A \times A$. Otherwise, for a given point x of A and any loop γ in A, the mapping $(s,t) \mapsto \zeta_s(x,\gamma(t))$ would be an homotopy between γ and the trivial loop $\{x\}$ and that would be a contradiction. Because of the stability axiom, the same topological argument may be used in the case where A is simply connected but ∂A is not.

A simple smoothness assumption ensures the existence.

Proposition 10 *Let D be a $\mathcal{C}^{k+1}$ bounded open set of $\mathbb{R}^n$ for a $k \geq 1$. There exists a k-regular interpolation mapping on $\overline{D}$.*

The following section is dedicated to the proof of that proposition.

2.3.2 Existence of an Interpolation Mapping: Proof

In the sequel, $k \in \mathbb{N}^*$ and D is a $\mathcal{C}^{k+1}$ bounded open set of $\mathbb{R}^n$. We denote by Γ its boundary.

To design the interpolation mapping ζ_s, we use the linear interpolation L_s when the arguments of ζ_s are "far from the boundary" and another mapping G_s when they are "near the boundary" in order to match the stability conditions. The general form of ζ_s is a suitable convex combination of these two mappings. The construction of G_s, more adapted to the boundary than the linear interpolation, is based on the description of a neighbourhood of Γ given in the following proposition.

Proposition 11 *There is an open neighbourhood U of Γ and a $\mathcal{C}^{k+1}$-diffeomorphism*

$$\varphi = (\varphi_t, \varphi_x) : U \to \,]-1;1[\, \times \Gamma$$

such that $\varphi_x(x) = x$ *if* $x \in \Gamma$ *and* $\varphi_t(x) \left| \begin{array}{ll} < 0 & \text{if } x \in D \cap U \\ = 0 & \text{if } x \in \Gamma \\ > 0 & \text{if } x \in \complement\overline{D} \cap U \end{array} \right.$

To prove this proposition, we will use the following lemma that asserts the existence of a smooth outgoing field to Γ:

Lemma 1 *There is a bounded $V \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that*

$$\forall\, x \in \Gamma,\ \langle V(x), n(x)\rangle_{\mathbb{R}^n} > 0 \tag{2.14}$$

Proof – The field V may be build by a suitable mollification of the gradient of the oriented distance function to the set D, defined by

$$b_D(x) = \begin{cases} d_D(x) & \text{if } x \in \complement D \\ -d_{\complement D}(x) & \text{if } x \in D \end{cases} \tag{2.15}$$

where d_D is the usual euclidean distance to D. The set D being a $\mathcal{C}^{k+1}$ open bounded set of $\mathbb{R}^n$ with $k \geq 1$, there exists $h > 0$ such that on the tubular neighbourhood U_h of Γ, given by

$$U_h = \{x \in \mathbb{R}^n,\ d_\Gamma(x) < h\} \tag{2.16}$$

b_D is of class $\mathcal{C}^{k+1}$. Moreover, ∇b_D is an extension of the unitary outer normal n to U_h and consequently

$$\langle \nabla b_D, n\rangle_{\mathbb{R}^n} = 1 \text{ on } \Gamma \tag{2.17}$$

(the reader is refered to [20] for further details). Let η be the standard mollifier (see [48]). For an arbitrary $\varepsilon > 0$, we set $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$ and $V_\varepsilon = \eta_\varepsilon * \nabla b_D$. Let O be an open neighbourhood of Γ which is compactly included in U_h. For $\varepsilon > 0$ small enough, V_ε belongs to $\mathcal{C}^\infty(O;\mathbb{R}^n)$ and converges uniformly to ∇b_D on O when ε tends to 0. Let $\varepsilon > 0$ be such that $\|V_\varepsilon - \nabla b_D\|_{\mathcal{C}^0(\overline{O})} < 1$. Then

$$\langle V_\varepsilon, n\rangle_{\mathbb{R}^n} \geq \langle \nabla b_D, n\rangle_{\mathbb{R}^n} - \|V_\varepsilon - \nabla b_D\|_{\mathcal{C}^0(\overline{O})} > 0$$

For any function $r \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R})$ such that $r|_\Gamma = 1$ and $r|_{\complement O} = 0$, the function

$$V(x) = \begin{cases} r(x)V_\varepsilon(x) & \text{if } x \in O \\ 0 & \text{otherwise.} \end{cases}$$

belongs to $\mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$, is bounded, and satisfies the inequality (2.14). ■

Now, we prove the proposition 11.

Proof– Thanks to the boundedness and regularity of the field V of the lemma 1, we define the flow $\phi : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ of the corresponding initial value problem

$$\begin{cases} \partial_t\phi(t,x) = V(\phi(t,x)) \\ \phi(0,x) = x \end{cases}$$

From the theory of ODEs, ϕ is a $\mathcal{C}^\infty$-mapping. Therefore, its restriction to Γ is a $\mathcal{C}^{k+1}$-mapping which is into. Indeed, for any $x \in \mathbb{R}^n$ the curve

$$C_x = \bigcup_{t\in\mathbb{R}}\{\phi(t,x)\}$$

crosses Γ *at most* once: if $y \in C_x \cap \Gamma$, then

$$C_x = C_y = C_y^- \cup \{y\} \cup C_y^+ \text{ with } C_y^- = \bigcup_{t<0}\{\phi(t,y)\} \text{ and } C_y^+ = \bigcup_{t>0}\{\phi(t,y)\}$$

and as V is outgoing from D

$$C_y^- \subset D \text{ and } C_y^+ \subset \complement\overline{D} \tag{2.18}$$

Therefore, if x and x' belong to Γ and $y = \phi(t,x) = \phi(t',x')$, x and x' belong to the same curve and to Γ: necessarily, $x' = x$. From the uniqueness of the Cauchy problem, we deduce that $t' = t$ (otherwise, $t \mapsto \phi(t,x)$ would be periodic and that but the inclusions (2.18) do not allow that case).

There is a $\tau > 0$ such that the restriction ψ of ϕ to $]-\tau;\tau[\times\Gamma$ is non-singular: Indeed, we have

$$\forall x \in \Gamma,\ \partial_t\psi(0,x) = V(x) \text{ and } \partial_x\psi(0,x) = I_{T_x\Gamma}$$

Therefore the tranversality assumption (2.14) implies that $D\psi(0,x)$ is full rank for any $x \in \Gamma$. By continuity of ψ and compacity of Γ, $D\psi$ is still full rank on $]-\tau;\tau[\times\Gamma$ for a small enough τ. If we replace V with $(1/\tau)V$, we obtain the same results with $\tau = 1$.

Then, the Inverse Mapping Theorem implies that ψ is a $\mathcal{C}^{k+1}$-diffeomorphism onto its open image U and as for any $x \in \Gamma$, $\psi(0,x) = x$, we have $\Gamma \subset U$. It is now obvious that $\varphi = \psi^{-1}$ satisfies the conclusions of the proposition 11. ■

We now prove the proposition 10. For any function $f : \mathbb{R}^m \to \mathbb{R}^p$ and $s \in [0,1]$, we define the associated interpolation operator $L_s(f)$ by

$$L_s(f)(x,y) = (1-s)f(x) + sf(y) \tag{2.19}$$

and we use simply the notation L_s when f is the identity.

Definition 3 *Let U be the open set given by the proposition 11 and let $h > 0$ be such that $U_{3h} \subset\subset U$. We set*

$$K_h = (\overline{U_{2h}} \times \overline{U_{2h}}) \cap \{(x,y) \in \mathbb{R}^n,\ |x-y| \leq h\} \tag{2.20}$$

and we define for any $s \in [0,1]$ the mapping $G_s : K_h \to \mathbb{R}^n$ by

$$G_s = \varphi^{-1} \circ (L_s(\varphi_t), \varphi_x \circ L_s) \tag{2.21}$$

or equivalently by

$$\varphi_t \circ G_s = L_s(\varphi_t) \ \text{ and } \ \varphi_x \circ G_s = \varphi_x \circ L_s \tag{2.22}$$

Remark – The mappings G_s may be defined on any set K such that

$$K \subset U \times U \ \text{ and } \ \forall s \in [0,1],\ L_s(K) \subset U \tag{2.23}$$

The set K_h obviously satisfies this property. □

We have clearly

$$\forall s \in [0,1], \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n,\ |L_s(x,y) - x| \leq |x-y| \tag{2.24}$$

An analogous property holds for the mappings G_s:

Lemma 17 *For any compact set $K \subset \mathbb{R}^n \times \mathbb{R}^n$ such that (2.23) holds, there is a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to 0} \omega(x) = 0$ and*

$$\forall s \in [0,1], \forall (x,y) \in K,\ |G_s(x,y) - x| \leq \omega(|x-y|) \tag{2.25}$$

Proof – This is merely a consequence of the uniform continuity of the function $(s,x,y) \mapsto G_s(x,y)$ on the compact $[0,1] \times K$ and of the equality $G_s(x,x) = x$ that holds for any $s \in [0,1]$. ■

Lemma 18 *For any $h > 0$, and any $\varepsilon \in]0, h/2[$, there is a function $\rho \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n; [0,1])$ such that for any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $|x-y| \leq \varepsilon$*

a) *if x or y belongs to $\complement D \cup \overline{U_h}$ then $\rho(x,y) = 0$.*

b) *if x or y belongs to $D - U_{2h}$ then $\rho(x,y) = 1$.*

Proof – If x or y belongs to $\complement D \cup \overline{U_h}$ and $|x-y| \leq \varepsilon$ then x and y are in $A = \complement D \cup \overline{U_{h+\varepsilon}}$. On the other hand, if x or y belongs to $D - U_{2h}$ then x and y are in $B = D - U_{2h-\varepsilon}$. Therefore the distance $d(A,B)$ is positive and we may build a function $\tilde{r} : \mathbb{R}^n \to [0,1]$ equal to 0 on a neighbourhood of A and 1 on a neighbourhood of B. A suitable mollification of $\tilde{r}$ provides a function $r \in \mathcal{C}^\infty(\mathbb{R}^n; [0,1])$ such that $r|_A = 0$ and $r|_B = 1$. The function $\rho = r \otimes r$ satisfies the properties of the lemma 18. ■

Let $h > 0$ be such that the tubular neighbourhood U_{3h} is compactly included in U. We set $K = K_h$ in the lemma 17 and choose a $\delta > 0$ such that $0 \leq \alpha \leq \delta$ implies $\omega(\alpha) \leq h$. We set $\varepsilon = \min(\delta, h/3)$.

Definition 4 *On the set* $\overline{X} = \{(x, y) \in \overline{D} \times \overline{D},\ |x - y| \leq \varepsilon\}$ *we define the mappings* ζ_s *for any* $s \in [0, 1]$ *by*

$$\zeta_s = \rho L_s + (1 - \rho) G_s \tag{2.26}$$

To conclude the proof of the proposition 10, we check that $[s \mapsto \zeta_s]$ is well defined and is a k-regular interpolation mapping on $\overline{D}$.

• The mappings ζ_s are well defined: indeed, if $(x, y) \in \overline{X}$, either $\rho(x, y) = 1$ or $G_s(x, y)$ is well-defined. It is a direct consequence of the property b) of the lemma 18 because if x and y are not in $D - U_{2h}$ then $(x, y) \in K_h$.

• Axiom (i): it is a direct consequence of the definition (2.21) and (2.26).

• Axiom (ii): for any $(x, y) \in \overline{X}$, either $\rho(x, y) = 0$ or x and y belong to $D - U_h$. In the first case, $\zeta_s(x, y) = G_s(x, y)$ and therefore, $G_s(x, y) \in \overline{D}$. In the other one, from the inequalities (2.24) and (2.25), we obtain $|\zeta_s(x, y) - x| \leq h$ and therefore $\zeta_s(x, y) \in \overline{D}$. That proves the first inclusion. If $(x, y) \in \overline{X} \cap \Gamma^2$, then $\zeta_s(x, y) = G_s(x, y)$ and as $\varphi_t(x) = \varphi_t(y) = 0$, $L_s(\varphi_t)(x, y) = 0$. Consequently, $\varphi_t(G_s(x, y)) = 0$ and $G_s(x, y) \in \Gamma$. The second inclusion is proved.

• Axiom (iii): the desired regularity of $[s \mapsto \zeta_s]$ comes from the fact that ϕ and its inverse are $\mathcal{C}^{k+1}$-mappings (see proposition 11).

2.4 Interpolation between diffeomorphisms

2.4.1 Definition of the functional interpolation

In this section, we define a functional interpolation between two transformations of $\overline{D}$. The properties of this interpolation will be detailed in the proposition 12.

To achieve that goal, we use the pointwise interpolation ζ described in the previous section and therefore this interpolation is only defined for close enough mappings θ_0 and θ_1.

Definition 5 *For any* θ_1 *and* θ_2 *in* $\mathrm{Diff}_k(\overline{D})$ *such that* $\mathrm{Range}(\theta_1, \theta_2) \subset \overline{X}$ *we define for any* $s \in [0, 1]$

$$T_s(\theta_0, \theta_1) = \zeta_s \circ (\theta_0, \theta_1) \tag{2.27}$$

We use simply the notation T_s for $T_s(\theta_1, \theta_2)$ if no confusion may occur.

Proposition 12 *For any* $\theta \in \mathrm{Diff}_k(\overline{D})$*, there exist* $\varepsilon > 0$ *and an increasing function* $\mu_k : \mathbb{R}_+ \to \mathbb{R}_+$ *which satisfies* $\lim_{x \to 0} \mu_k(x) = 0$ *such that for any* θ_0 *and* θ_1 *in* $B_k(\theta, \varepsilon)$*, the following properties hold.*

i) Interpolation: $T_0(\theta_0, \theta_1) = \theta_0$ *and* $T_1(\theta_0, \theta_1) = \theta_1$*. Moreover if* $\theta_0 = \theta_1$ *then for all* $s \in [0, 1]$*,* $T_s(\theta_0, \theta_1) = \theta_0 = \theta_1$*.*

ii) Stability: *for any* $s \in [0, 1]$*,* $T_s(\theta_0, \theta_1)$ *is a* $\mathcal{C}^k$*-diffeomorphism from* $\overline{D}$ *to* $\overline{D}$*.*

iii) Regularity: $s \mapsto T_s(\theta_0, \theta_1)$ *belongs to* $\mathcal{C}^1([0, 1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^N))$ *and for any* $s \in [0, 1]$*,* $\|\partial_s T_s(\theta_0, \theta_1)\|_k \leq \mu_k(\|\theta_0 - \theta_1\|_k)$*.*

The following lemma simplifies the proof of the previous proposition.

Lemma 2 *Assume that the proposition 12 holds with* $\theta = I$*. Then it also holds for any* $\theta \in \mathrm{Diff}_k(\overline{D})$*.*

Proof – We assume that the proposition 12 holds with $\theta = I$ and we choose the corresponding value of ε. Let θ belong to $\mathrm{Diff}_k(\overline{D})$. Then, there is a $\varepsilon' > 0$ such that $\theta' \in B_k(\theta, \varepsilon')$ implies $\theta' \circ \theta^{-1} \in B_k(I, \varepsilon)$. Indeed, let $A_\theta : \mathcal{C}^k(\overline{D}; \mathbb{R}^n) \to \mathcal{C}^k(\overline{D}; \mathbb{R}^n)$ be the linear continuous operator given by $A_\theta(S) = S \circ \theta^{-1}$. Then

$$\|\theta' \circ \theta^{-1} - I\|_k = \|(\theta' - \theta) \circ \theta^{-1}\|_k \leq \|A_\theta\| \cdot \varepsilon$$

and therefore the choice of $\varepsilon' = \varepsilon / \|A_\theta\|$ is adequate.

Let θ_0 and θ_1 belong to $B_k(\theta, \varepsilon')$. For any such mappings, the definition (2.27) of T_s yields

$$T_s(\theta_0, \theta_1) = T_s(\theta_0 \circ \theta^{-1}, \theta_1 \circ \theta^{-1}) \circ \theta \tag{2.28}$$

of which properties *i)* and *ii)* are easy consequences. The regularity required in the point *iii)* is obvious. Moreover, the equation (2.28) also yields

$$\begin{aligned} \|\partial_s T_s(\theta_0, \theta_1)\|_k &= \|\partial_s T_s(\theta_0 \circ \theta^{-1}, \theta_1 \circ \theta^{-1}) \circ \theta\|_k \\ &\leq \|A_\theta\| \cdot \mu_k(\|A_\theta\| \|\theta_0 - \theta_1\|_k) \\ &\leq \bar{\mu}_k(\|\theta_0 - \theta_1\|_k) \end{aligned}$$

with $\bar{\mu}_k(x) = \|A_\theta\| \cdot \mu(\|A_\theta\| x)$. As $\bar{\mu}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function such that $\bar{\mu}_k(x) \to 0$ when $x \to 0$, the property *iii)* is proved in the general case. ■

In the case $\theta = I$, the property $i)$ is a mere consequence of axiom (i) of the construction of ζ. The proof of property $ii)$, rather technical, is postponed to the next subsection. The assumption (iii) for k-regular interpolation mappings obviously yields $s \mapsto T_s(\theta_0, \theta_1) \in \mathcal{C}^1([0,1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^N))$. The existence of the function μ is the purpose of the following lemma.

Before its statement, we introduce some tensorial and differentiel notations and we recall the composite mapping formula.

• **k-th order differential.** Let U be an open subset of $\mathbb{R}^n$ and F a Banach space. For any $k \in \mathbb{N}$, $D^k f$ is the k-th order differential of the mapping $f : U \to F$, given by

$$D^k f \cdot (h_1, ..., h_k) = \sum_{i \in \{1,...,N\}^k} \frac{\partial f}{\partial x_{i_1} ... \partial x_{i_k}} \cdot h_1^{i_1} \times ... \times h_k^{i_k}$$

• **Tensorial conventions.** For any n-linear mapping A on $E_1 \times ... \times E_n$ and any i_n-linear mappings B_1, ..., B_n with values in E_1, ..., E_n such that $i_1 + ... + i_n = k$, we define the k-linear mapping $A \cdot (B_1, ..., B_n)$ by

$$[A \cdot (B_1, ..., B_n)](h_1, ..., h_k) = A(B_1(h_1, ..., h_{i_1}), ..., B_n(h_{k-i_n+1}, ..., h_k)) \tag{2.29}$$

• **Composite mapping formula.** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be some open subsets and F be a Banach space. The composition of the $\mathcal{C}^r$ mappings $g : U \to V$ and $f : V \to F$ is of class $\mathcal{C}^r$ and for any $k \in \{1, ..., r\}$, we have

$$D^k(f \circ g) = \sum_{q=1}^{k} \sum_{i \in I(q,k)} \sigma_k(i) \cdot (D^q f) \circ g \cdot (D^{i_1} g, ..., D^{i_q} g) \tag{2.30}$$

where for any $q \in \mathbb{N}$ and $k \in \mathbb{N}$

$$I(q,k) = \{(i_1, ..., i_q) \in (\mathbb{N}^*)^q, \ |i| = i_1 + \cdots + i_q = k\} \tag{2.31}$$

(for $q = k = 0$, we have $I(0,0) = \{\epsilon\}$ where ϵ is the unique mapping from $\varnothing$ to $\mathbb{N}^*$ and for $k \in \mathbb{N}^*$, $I(0,k) = \varnothing$). For any $k \in \mathbb{N}$, the mappings σ_k

$$\sigma_k : I_k = \bigcup_{q=0}^{k} I(q,k) \to \mathbb{N} \tag{2.32}$$

are defined by $\sigma_0(\epsilon) = 1$ and

$$\forall\, k \in \mathbb{N},\ \forall\, q \in \{0, ..., k\},\ \forall\, i \in I_q,\ \sigma_{k+1}(i, k-q+1) = \binom{k}{q} \sigma_q(i) \tag{2.33}$$

(see for example [1], prop. 1.4, p. 3). By continuity, the formula (2.30) still holds when $g \in \mathcal{C}^r(\overline{U}; \mathbb{R}^m)$ and $f \in \mathcal{C}^r(\overline{V}; \mathbb{R}^n)$.

Lemma 3 *There exists a $\varepsilon > 0$ and an increasing function $\mu_k : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x\to 0} \mu_k(x) = 0$ such that, for any θ_0 and θ_1 in $B_k(\theta, \varepsilon)$*

$$\|\partial_s T_s(\theta_0, \theta_1)\|_k \leq \mu_k(\|\theta_0 - \theta_1\|_k)$$

Proof –

• For $k = 0$, since ζ satisfies the regularity axiom (iii), $\partial_s \zeta_s$ belongs to $\mathcal{C}^0([0,1], \mathcal{C}^k(\overline{X}; \mathbb{R}^N))$. Accordingly the mapping $(s, x, y) \mapsto \partial_s \zeta_s(x, y)$ defined on the compact set $[0,1] \times \overline{X}$ is continuous so it has a modulus of continuity μ. Since $\zeta_s \circ (\theta_0, \theta_0) = \theta_0$, we have

$$\partial_s T_s = \partial_s \zeta_s \circ (\theta_0, \theta_1) = \partial_s \zeta_s \circ (\theta_0, \theta_1) - \partial_s \zeta_s \circ (\theta_0, \theta_0)$$

so we end up with

$$\|\partial_s T_s\|_0 \leq \mu(\|\theta_0 - \theta_1\|_0)$$

as desired.

• For $k \geq 1$, we need to estimate the (spatial) derivates of $\partial_s T_s$.

From now on, we assume that we have already proved the existence of increasing mappings $\mu_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu_n(x) \to 0$ when $x \to 0$ and

$$\text{for any } n \in \{0, ..., k-1\},\ \|D^n(\partial_s T_s)\|_0 \leq \mu_n(\|\theta_0 - \theta_1\|_n) \tag{2.34}$$

The composite mapping formula asserts that the k-th order differential of $\partial_s T_s = \partial_s \zeta_s \circ (\theta_0, \theta_1)$ has the form

$$D^k(\partial_s T_s) = \sum_{q=1}^{k} \sum_{i \in I(q,k)} \sigma_k(i) [D^q(\partial_s \zeta_s) \circ (\theta_0, \theta_1)] \cdot (D^{i_1}(\theta_0, \theta_1), ..., D^{i_q}(\theta_0, \theta_1)) \tag{2.35}$$

As a consequence of the interpolation axiom for ζ, $\partial_s \zeta_s(\theta_0, \theta_0)$ is identically equal to zero on $\overline{D}$ as its k-th order differential. Therefore, we have

$$\sum_{q=1}^{k} \sum_{i \in I(q,k)} \sigma_k(i) [D^q(\partial_s \zeta_s) \circ (\theta_0, \theta_0)] \cdot (D^{i_1}(\theta_0, \theta_0), ..., D^{i_q}(\theta_0, \theta_0)) = 0 \tag{2.36}$$

Using that equation, we end up with

$$\begin{aligned}
D^k(\partial_s T_s) &= \sum_{q=1}^{k} \sum_{i \in I(q,k)} \sigma_k(i)[D^q(\partial_s \zeta_s) \circ (\theta_0, \theta_1)] \cdot (D^{i_1}(\theta_0,\theta_1), ..., D^{i_q}(\theta_0,\theta_1)) \\
&= \sum_{q=1}^{k} \sum_{i \in I(q,k)} \sigma_k(i)[D^q(\partial_s \zeta_s) \circ (\theta_0, \theta_1) - D^q(\partial_s \zeta_s) \circ (\theta_0, \theta_0)] \cdot (D^{i_1}(\theta_0,\theta_1), ..., D^{i_q}(\theta_0,\theta_1)) \\
&\quad + \sum_{q=1}^{k} \sum_{i \in I(q,k)} \sigma_k(i)[D^q(\partial_s \zeta_s) \circ (\theta_0, \theta_0)] \cdot (D^{i_1}(0,\theta_1 - \theta_0), ..., D^{i_q}(\theta_0,\theta_1)) \\
&\quad + \sum_{q=1}^{k} \sum_{i \in I(q,k)} \sigma_k(i)[D^q(\partial_s \zeta_s) \circ (\theta_0, \theta_0)] \cdot (D^{i_1}(\theta_0,\theta_0), D^{i_2}(0,\theta_1 - \theta_0)..., D^{i_q}(\theta_0,\theta_1)) \\
&\quad + \cdots \\
&\quad + \sum_{q=1}^{k} \sum_{i \in I(q,k)} \sigma_k(i)[D^q(\partial_s \zeta_s) \circ (\theta_0, \theta_0)] \cdot (D^{i_1}(\theta_0,\theta_0), D^{i_2}(\theta_0,\theta_0)..., D^{i_q}(0,\theta_1 - \theta_0))
\end{aligned}$$

and if ω is a modulus of continuity of the mappings $(s, x, y) \mapsto D^q(\partial_s \zeta_s(x, y))$ on $[0, 1] \times \overline{X}$ for a $1 \leq q \leq k$, then, for adequate constants K_1 and K_2, we have

$$\|D^k(\partial_s T_s)\|_0 \leq K_1 \cdot \omega(\|\theta_0 - \theta_1\|_0) + K_2 \cdot \|\theta_0 - \theta_1\|_k \tag{2.37}$$

which, together with (2.34), proves the lemma. ■

2.4.2 Proof of the stability property

In this section, we prove the point *ii*) of the proposition 12 in the special case $\theta = I$. The lemma 2 shows that this is enough to prove that this proposition holds for any $\theta \in \mathrm{Diff}_k(\overline{D})$.

Lemma 4 *There exists $\varepsilon > 0$ such that, for any s in $[0, 1]$, $\theta_0 \in B_k(I, \varepsilon)$ and $\theta_1 \in B_k(I, \varepsilon)$, $T_s(\theta_0, \theta_1)$ is a non-singular transformation on $\overline{D}$* i.e

$$\forall\, x \in \overline{D},\ DT_s(x) \ \text{ is invertible} \tag{2.38}$$

Proof – For a small enough ε, $\mathrm{Range}(\theta_1, \theta_2)$ is included in $\overline{X}$. Consequently $T_s(\theta_0, \theta_1)$ is well-defined on $\overline{D}$ for any $s \in [0, 1]$, with values in $\mathcal{C}^k(\overline{D}, \mathbb{R}^n)$.For any $s \in [0, 1]$ we have

$$DT_s = \partial_x \zeta_s \circ (\theta_0, \theta_1) \cdot D\theta_0 + \partial_y \zeta_s \circ (\theta_0, \theta_1) \cdot D\theta_1$$

Due to the interpolation axiom (i), $\partial_x \zeta_s + \partial_y \zeta_s = I$ on $\overline{X}$. Thus

$$\begin{aligned}
DT_s - I_n \quad = \quad & \partial_x[\zeta_s \circ (\theta_0, \theta_1) - \zeta_s \circ (I, I)] \cdot D\theta_0 \\
& + \partial_y[\zeta_s \circ (\theta_0, \theta_1) - \zeta_s \circ (I, I)] \cdot D\theta_1 \\
& + \partial_x \zeta_s \circ (I, I) \cdot [D\theta_0 - I_n] + \partial_y \zeta_s \circ (I, I) \cdot [D\theta_1 - I_n]
\end{aligned}$$

where I_n is the $n \times n$ identity matrix. Since $(s, x, y) \mapsto \zeta_s(x, y) \in \mathcal{C}^1([0, 1] \times \overline{X})$, the mappings $(s, x, y) \mapsto \partial_x \zeta(s, x, y)$ and $(s, x, y) \mapsto \partial_y \zeta(s, x, y)$ are uniformly continuous on $[0, 1] \times \overline{X}$. It follows that there exists $\varepsilon > 0$ such that

$$\|\theta_0 - I\|_1 < \varepsilon \ \text{ and } \ \|\theta_1 - I\|_1 < \varepsilon \ \Rightarrow \ \|DT_s - I_n\|_0 < 1 \tag{2.39}$$

Hence for any such ε and for any $s \in [0, 1]$ and $x \in \overline{D}$, $DT_s(x)$ is invertible. ■

Lemma 5 *There exists $\varepsilon > 0$ such that, for any s in $[0,1]$, $\theta_0 \in B_k(I,\varepsilon)$ and $\theta_1 \in B_k(I,\varepsilon)$, $T_s(\theta_0,\theta_1)$ is a one-to-one correspondance from $\overline{D}$ to $\overline{D}$.*

Proof – a) The mapping T_s is into.

Let $\varepsilon_0 > 0$ be such that the lemma 4 is satisfied with $\varepsilon = \varepsilon_0$. Let $r > 0$ be such that $\overline{D} \subset B(0,r)$. There is a linear continuous extension operator

$$E : \mathcal{C}^1(\overline{D};\mathbb{R}^n) \to \mathcal{C}^1(\overline{B}(0,r);\mathbb{R}^n) \tag{2.40}$$

For any θ_0 and θ_1 in $B_k(I,\varepsilon_0)$, we define the extension

$$\hat{T}_s : \overline{B}(0,r) \to \mathbb{R}^n \tag{2.41}$$

of T_s by

$$\hat{T}_s = I_{\overline{B}(0,r)} + E(T_s - I_{\overline{D}})$$

As in the proof of the lemma 4, for any $\eta > 0$, we may found a $\varepsilon(\eta) \in (0,\varepsilon_0]$ such that $\theta_0 \in B_k(I,\varepsilon(\eta))$ and $\theta_1 \in B_k(I,\varepsilon(\eta))$ yield $\|T_s - I_{\overline{D}}\|_1 < \eta$.

For any x and y in $\overline{D}$, we have

$$T_s(y) - T_s(x) = \hat{T}_s(y) - \hat{T}_s(x) = \int_0^1 D\hat{T}_s(x + t(y-x)) \cdot (y-x)\,dt$$

Accordingly

$$T_s(y) - T_s(x) = y - x + \int_0^1 [D\hat{T}_s(x + t(y-x)) - I_n] \cdot (y-x)\,dt$$

which yields

$$|(T_s(y) - T_s(x)) - (y-x)| \leqslant \|DT_s - I_n\|_0 |x-y| \leq \|E\| \cdot \|T_s - I_{\overline{D}}\|_1 \cdot |x-y|$$

and finally if $\theta_0 \in B_k(I,\varepsilon(\eta))$ and $\theta_1 \in B_k(I,\varepsilon(\eta))$ with $\eta = \dfrac{1}{2\|E\|}$, we obtain

$$\frac{|x-y|}{2} \leqslant |T_s(y) - T_s(x)| \tag{2.42}$$

and this equation implies that T_s is into.

b) The mapping T_s is onto.

The proof rely on the following lemma :

Let B be an open subset of $\mathbb{R}^n$ and B_i, for $i \in I$, its connected components. For any open subset A of B such that for any $i \in I$, $A_i = A \cap B_i \neq \varnothing$

$$A = B \text{ if and only if } \forall\, i \in I,\ \partial A_i \subset \partial B_i \tag{2.43}$$

The proof is made by contradiction. Assume that for any $i \in I$, $\partial A_i \subset \partial B_j$ and that there exists $x \in B - A$. Let $j \in I$ be such that $x \in B_j - A_j$. Since B_j is open and connected, for any $y \in A_j$, there exists a continuous path $\lambda : [0,1] \to B_j$ such that $\lambda(0) = x$ and $\lambda(1) = y$ and therefore there is a $t \in [0,1]$ such that $\lambda(t) \in \partial A_j$. That contradicts the inclusion $\partial A_j \subset \partial B_i \subset \complement B_j$. The converse implication is obvious.

Let D_i, $i \in I$, be the connected components of D. Under the assumptions of the lemma 4, the restriction of T_s to any D_i is a local $\mathcal{C}^k$-diffeomorphism. Therefore, its image is open in $\mathbb{R}^n$ and included in $\overline{D}$, hence in D. There is a $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$\theta_0 \in B_k(I, \varepsilon) \text{ and } \theta_1 \in B_k(I, \varepsilon) \Rightarrow \|T_s - I_n\|_0 < \min_{i \neq j} d(D_i, D_j)$$

For any such ε, $T_s(D_i) \subset D_i$ and $T_s(\overline{D_i}) \subset \overline{D_i}$ for any $i \in I$ and consequently, $T_s(\partial D_i) \subset \partial D_i$.

Let y be a point of $\partial T_s(D_i)$ and y_n, $n \in \mathbb{N}$, a sequence of $T_s(D_i)$ which converges towards y. As a consequence of the compactness of $\overline{D_i}$, there is a sequence $x_n \in D_i$ and $x \in \overline{D_i}$ such that $T_s(x_n)$ is a subsequence of the y_n, $n \in \mathbb{N}$ and $x_n \to x$. By continuity, $T_s(x) = y$.

As $T_s(D_i) \subset D_i$, $x \in \partial D_i$. This proves $\partial T_s(D_i) \subset T_s(\partial D_i) \subset \partial D_i$. Using (2.43) with $A = T_s(D)$ and $B = D$, we come to $T_s(D) = D$ and $T_s(\partial D) = \partial D$. This provides T_s is onto from $\overline{D}$ to $\overline{D}$. ■

Lemma 6 *There exists $\varepsilon > 0$ such that, for any s in $[0,1]$, $\theta_0 \in B_k(I, \varepsilon)$ and $\theta_1 \in B_k(I, \varepsilon)$, $[T_s(\theta_0, \theta_1)]^{-1} \in \mathcal{C}^k(\overline{D}, \mathbb{R}^n)$.*

Proof – We take a $\varepsilon > 0$ such that the lemmas 4 and 5 are satisfied. For $r > 0$, we set $D_r = \overline{D} \oplus B(0, r) = \{x \in \mathbb{R}^n,\, d(x, D) < r\}$. Let $\hat{T}_s$ be a $\mathcal{C}^k$-extension of T_s to $\mathbb{R}^n$. There is a $r_0 > 0$ such that the restriction of $\hat{T}_s$ to D_{r_0} is into. If this property didn't hold, there would exist a sequence (x_n, y_n) in $\mathbb{R}^n \times \mathbb{R}^n$ with $x_n \neq y_n$, $\hat{T}_s(x_n) = \hat{T}_s(y_n)$, $d(x_n, D) \to 0$ and $d(y_n, D) \to 0$. Hence, up to passing to a subsequence, (x_n, y_n) has a limit $(x, y) \in \overline{D} \times \overline{D}$. Since $\hat{T}_s$ is continuous, $\hat{T}_s(x) = \hat{T}_s(y)$ and therefore $x = y$ but that would contradict the inversibility of $DT_s(x)$ that implies that $\hat{T}_s$ is a local diffeomorphism around x.

By continuity of DT_s, there is a $r_1 > 0$ such that $D\hat{T}_s$ is invertible on D_{r_1}. The restriction of $\hat{T}_s$ to D_r with $r = \min(r_0, r_1)$ is a $\mathcal{C}^k$-diffeomorphism onto its open image. This image contains $\overline{D}$ and therefore T_s^{-1} belongs to $\mathcal{C}^k(\overline{D}; \mathbb{R}^n)$. ■

With this lemma, we achieve the proof of $T_s(\theta_0, \theta_1)$ being a $\mathcal{C}^k$-diffeomorphism of $\overline{D}$, which achieve the proof of proposition 12.

2.5 Comparison of the limits and applications

2.5.1 Prerequisite: a Lemma on Summable Series

Lemma 7 *Let $(\sigma_n)_{n \in \mathbb{N}}$ be a non-negative sequence such that*

$$\sum_{n=0}^{+\infty} \sigma_n < +\infty$$

Then, there exist a non-negative sequence $(\delta_n)_{n \in \mathbb{N}}$ such that

$$\sigma_n = \mathrm{o}(\delta_n) \text{ and } \sum_{n=0}^{+\infty} \delta_n < +\infty$$

Remark – We notice that the sequences involved in the preceeding lemma are such that for n large enough, $\delta_n = 0$ implies $\sigma_n = 0$. The contruction of $(\delta_n)_{n\in\mathbb{N}}$ that follows implies in fact that this property is true *for any* $n \in \mathbb{N}$.

The lemma being proved, we may obviously assume that the sequence $(\delta_n)_{n\in\mathbb{N}}$ satisfies $\sum_{n=0}^{+\infty} \delta_n = 1$. □

Proof – We set $S_N = \sum_{n=0}^{N} \sigma_n$ and $S = \lim_{N\to+\infty} S_N$. Let $f : [0, S] \mapsto [0, 1]$ be an increasing mapping such that $f(0) = 0$, $f(S) = 1$, which is continuous at $x = S$. We assume moreover that $f \in \mathcal{C}^1([0, S[; \mathbb{R})$ and that $f'(x) \to +\infty$ when $x \to S^-$.

We set $\delta_0 = 0$ and $\delta_n = f(S_n) - f(S_{n-1})$ for $n \geq 1$. Because f is increasing, $(\delta_n)_{n\in\mathbb{N}}$ is non-negative and $\delta_n = 0$ iff $\sigma_n = 0$. Moreover, for any $\sigma_n > 0$,

$$\delta_n = \frac{f(S_{n-1} + \sigma_n) - f(S_{n-1})}{\sigma_n} \geq \inf_{[S_{n-1};1[} f' \to +\infty \text{ when } n \to +\infty$$

Consequently, $\sigma_n = \mathrm{o}(\delta_n)$. As desired, $\sum_{n=0}^{N} \delta_n = f(S_N) - f(S_0)$ and therefore $\sum_{n=0}^{+\infty} \delta_n = 1$. ■

2.5.2 Bounded μ-Variation Sequences

Definition 6 *Let μ be an increasing function $\mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to0} \mu(x) = 0$. A sequence $(L_n)_{n\in\mathbb{N}}$ of $\mathrm{Diff}_k(\overline{D})$ is of* bounded μ-variation *if*

$$\sum_{n=0}^{+\infty} \mu(\|L_{n+1} - L_n\|_k) < +\infty \tag{2.44}$$

Proposition 13 *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of $\mathrm{Diff}_k(\overline{D})$ of bounded μ_k-variation such that $L_n \to L$ in $\mathrm{Diff}_k(\overline{D})$. There is a $n_0 \in \mathbb{N}$ and a mapping $s \mapsto \theta_s \in \mathcal{C}^1([0,1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$ such that:*

i) $\forall\, s \in [0, 1]$, $\theta_s \in \mathrm{Diff}_k(\overline{D})$.

ii) there is a non-increasing sequence $(s_n)_{n\in N}$ with

$$\lim_{n\to+\infty} s_n = 0 \text{ and } \forall\, n \geq n_0,\, \theta_{s_n} = L_n$$

Proof – We set $\theta =$ in the proposition 12 and take the corresponding value of ε. There is a $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $L_n \in B_k(T, \varepsilon)$. We define the mapping $s \mapsto R_s \in \mathcal{C}^0([n_0, +\infty); \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$ by

$$\forall s \in [n; n+1],\ R_s = T_{s-n}(L_n, L_{n+1}) \tag{2.45}$$

The mapping $s \mapsto R_s$ belongs to $\mathcal{C}^1([n, n+1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$ for any $n \geq n_0$. Moreover,

$$\forall\, s \in (n, n+1),\ \|\partial_s R_s\|_k \leq \mu_k(\|L_{n+1} - L_n\|) \tag{2.46}$$

We introduce a $\mathcal{C}^1$-function $\omega : [0, 1] \to [0, 1]$ such that $\omega(0) = 0$, $\omega(1) = 1$ and $\omega'(0) = \omega'(1) = 0$. We denote $\kappa = \max_{[0,1]} \omega'$ and define ω_a^b for any $a < b$ by

$$\omega_a^b(x) = (b-a) \cdot \omega\left(\frac{x-a}{b-a}\right) + a$$

Then, we associate to $s \mapsto R_s$ the mapping $s \mapsto S_s$ such that

$$\forall n \geq n_0,\ \forall s \in [n, n+1],\ S_s = R_{\omega_n^{n+1}(s)}$$

From the properties of ω and $s \mapsto R_s$, it is clear that the mapping $s \mapsto S_s \in \mathcal{C}^1([n_0, +\infty); \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$, that for any $s \in [n_0, +\infty)$ S_s belongs to $\mathrm{Diff}_k(\overline{D})$, that for any $n \geq n_0$, $S_n = L_n$ and that

$$\forall\, s \in (n, n+1),\ \|\partial_s S_s\|_k \leq \kappa \mu_k(\|L_{n+1} - L_n\|) \tag{2.47}$$

Finally, let s_n be a non-increasing sequence (to be chosen later) such that

$$s_{n_0} = 1,\ \lim_{n \to +\infty} s_n = 0 \ \text{ and } \ L_{n+1} \neq L_n \Rightarrow s_{n+1} < s_n \tag{2.48}$$

We define the mapping $\theta : s \in (0,1] \to \mathcal{C}^k(\overline{D}; \mathbb{R}^n)$ by

$$\forall \lambda \in [0,1],\ \forall n \in \mathbb{N},\ \theta_{\lambda s_n + (1-\lambda) s_{n+1}} := S_{n+\lambda}$$

Obviously, for any sequence s_n, $\theta \in \mathcal{C}^1((0,1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$, for any $s \in [0,1]$, θ_s belongs to $\mathrm{Diff}_k(\overline{D})$ and for any $n \geq n_0$, $\theta_{s_n} = L_n$.

We show now that θ may be extended in a $\mathcal{C}^1([0,1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$ mapping for an adequate choice of the sequence s_n by proving that $\partial_s \theta_s \to 0$ when $s \to 0^+$.

Clearly, for any $s \in [s_{n+1}, s_n]$ (with $s_{n+1} \neq s_n$), we have

$$\|\partial_s \theta_s\|_k \leqslant \frac{1}{s_n - s_{n+1}} \cdot \|\partial_s S_s\|_k \leqslant \frac{\kappa}{s_n - s_{n+1}} \cdot \mu_k(\|L_{n+1} - L_n\|_k) \tag{2.49}$$

We define the sequence $(\sigma_n)_{n \in \mathbb{N}}$ by

$$\sigma_n = \mu_k(\|L_{n+1} - L_n\|_k) \tag{2.50}$$

It satisfies the assumptions of the lemma 7. We set $s_{n_0} = 1$ and $s_{n+1} = s_n - \delta_n$, where $(\delta_n)_{n \in \mathbb{N}}$ is given by this lemma (and the subsequent remark). Clearly, $(s_n)_{n \in \mathbb{N}}$ satisfies the properties (2.48) and for any $n \geq n_0$ such that $s_{n+1} < s_n$ and any $s \in [s_{n+1}, s_n]$

$$\|\partial_s \theta_s\|_k \leq \kappa \cdot \frac{\sigma_n}{\delta_n} \to 0 \ \text{ when } \ n \to +\infty$$

The proof is completed. ∎

2.5.3 $\mathcal{C}^1$-paths in $\mathrm{Diff}_k(\overline{D})$ and Velocity Fields

Proposition 14 *Let $s \mapsto L_s : [0,1] \to \mathcal{C}^k(\overline{D}; \mathbb{R}^n)$ such that $L_0 = I$. The following properties are equivalent*

(i) *$s \mapsto L_s$ is a $\mathcal{C}^1$-path of $\mathrm{Diff}_k(\overline{D})$.*

(ii) *there is a $V \in \mathcal{V}_k(D)$ such that for any $s \in [0,1]$, $L_s = T_s(V)$.*

This result provides a clear characterization of the trajectories that may be generated as a the flow of a vector field. It is classical with the extra assumption that $s \mapsto T_s^{-1}$ is continuous (see [47] th. 2.16, p. 51). We prove that extra property in this subsection.

Lemma 8 *Let E and F be two Banach spaces and $U \subset E$, $V \subset F$ two open subsets. The mappings $\mathsf{C} : \mathcal{C}^0(\overline{V}; \mathbb{R}) \times \mathcal{C}^0(\overline{U}; \overline{V}) \to \mathcal{C}^0(\overline{U}; \mathbb{R})$ and $\mathsf{P} : \mathcal{C}^0(\overline{U}; \mathbb{R}) \times \mathcal{C}^0(\overline{U}; \mathbb{R}) \to \mathcal{C}^0(\overline{U}; \mathbb{R})$ defined by*

$$\mathsf{C}(f,g) = f \circ g \ \text{ and } \ \mathsf{P}(f,g) = fg \tag{2.51}$$

are continous.

Proposition 15 *Let $s \mapsto L_s : [0,1] \to$ be a $\mathcal{C}^1$-path of $\mathrm{Diff}_k(\overline{D})$. The mapping $s \mapsto L_s^{-1}$ belongs to $\mathcal{C}^0([0,1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$.*

Proof – Step 1 : the mapping $s \mapsto L_s^{-1}$ belongs to $\mathcal{C}^0([0,1]; \mathcal{C}^0(\overline{D}; \mathbb{R}^n))$.

For any $(s,t) \in [0,1]^2$, we have

$$\begin{aligned} \|L_t^{-1} - L_s^{-1}\|_0 &= \|I - L_s^{-1} \circ L_t\|_0 = \|L_s^{-1} \circ L_s - L_s^{-1} \circ L_t\|_0 \\ &\leq \|[DL_s]^{-1}\|_0 \cdot \|L_t - L_s\|_0 \to 0 \text{ when } t \to s \end{aligned}$$

Step 2 : the mapping $s \mapsto L_s^{-1}$ belongs to $\mathcal{C}^0([0,1]; \mathcal{C}^1(\overline{D}; \mathbb{R}^n))$.

Both L_s^{-1} and DL_s are continuous with respect to s. Therefore, the lemma 8 yields the continuity of $s \mapsto DL_s \circ L_s^{-1}$. For any compact set K included in the set of $n \times n$ invertible matrices, the mapping $\mathrm{Inv} : K \to \mathbb{R}^{n \times n}$ defined by $\mathrm{Inv}(A) = A^{-1}$ is uniformly continuous and therefore, $DL_s^{-1} = [DL_s \circ L_s^{-1}]^{-1}$ is continuous with respect to s.

Step 3 : the mapping $s \mapsto L_s^{-1}$ belongs to $\mathcal{C}^0([0,1]; \mathcal{C}^l(\overline{D}; \mathbb{R}^n))$ for any $l \in \{0, ..., k\}$.

We assume that this result holds for any $l \in \{0, ..., r-1\}$ with $2 \leq r \leq k$. Then, for any $s \in [0,1]$, we set $f_s = L_s$ and $g_s = L_s^{-1}$ and we use the composite mapping formula (2.30). The definition (2.33) implies that $\forall\, r \in \mathbb{N}^*$, $\sigma_r(r) = 1$. Moreover, for any $s \in [0,1]$ and $r \geq 2$, $D^r(f_s \circ g_s) = 0$. Therefore, we have

$$D^r g_s = -[(Df_s) \circ g_s]^{-1} \cdot \sum_{q=2}^{r} \sum_{i \in I(q,r)} \sigma_r(i) \cdot (D^q f_s) \circ g_s \cdot (D^{i_1} g_s, ..., D^{i_q} g_s) \quad (2.52)$$

For any $q \geq 2$ and any $i = (i_1, ..., i_q) \in I(q,r)$, $\forall\, j \in \{1, ..., q\}$, $i_j \geq 1$ and $i_1 + \cdots + i_q = r$. Therefore, $\forall\, j \in \{1, ..., q\}$, $i_j \leq r-1$: the orders of differentiation of g in the right-hand side of (2.52) are lower than $r-1$. Therefore, all the $D^{i_j} g_s$ involved in the right-hand side of (2.52) are continuous with respect to s. We have already shown that $s \mapsto [(Df_s) \circ g_s]^{-1}$ was continuous and $s \mapsto (D^q f_s) \circ g_s$ is continuous as the composition of continuous mappings. Hence, $s \mapsto D^r g_s$ is continuous as the product of continuous mappings (see the lemma 8). ■

2.5.4 Continuity of Shape Functionals

We state the main result of the paper

Theorem 1 *The directional continuity and the (classical) continuity of a shape functional $J : \mathcal{A} \to X$ are equivalent.*

To show, this, we first prove the following elementary lemma:

Lemma 9 *From any converging sequence $(L_n)_{n \in \mathbb{N}}$ of $\mathrm{Diff}_k(\overline{D})$, we may extract a bounded μ_k-variation subsequence.*

Proof – Let L be the limit of the sequence. The function μ_k is increasing, $\lim_{x \to 0} \mu_k(x) = 0$, and the sequence $\|L_n - L\|_k$ tends to 0 when $n \to +\infty$. We define the non-decreasing sequence $(i_n)_{n \in \mathbb{N}}$ by

$$i_n = \min \left\{ i \in \mathbb{N},\ \forall\, p \geq i,\ \mu_k(\|L_{p+1} - L_p\|_k) \leq \frac{1}{1+n^2} \right\} \quad (2.53)$$

We may then define the increasing sequence $(j_n)_{n\in\mathbb{N}}$ by

(i) removing from the sequences the values of $(i_n)_{n\in\mathbb{N}}$ that are obtained twice or more.

(ii) concatenating with any subsequence that makes the $(j_n)_{n\in\mathbb{N}}$ increasing if the set $\{j_n,\, n\in\mathbb{N}\} = \{i_n,\, n\in\mathbb{N}\}$ is finite (that is if $L_n = L$ for $n \geq n_0$).

Then the sequence of L_{j_n}, $n\in\mathbb{N}$ is of bounded μ_k-variation as

$$\sum_{n=0}^{+\infty} \mu_k(\|L_{j_n+1} - L_{j_n}\|_k) \leq \sum_{n=0}^{+\infty} \frac{1}{1+n^2} < +\infty$$

■

We may now prove the theorem 1:

Proof – The continuity of J obviously implies its directional continuity. Let $J : \mathcal{A} \to \mathbb{R}$ be a shape functional that is directionaly continuous at $\Omega \in \mathcal{A}$. We define $j : \mathrm{Diff}_k(\overline{D}) :\to X$ by

$$j(\theta) := J(\Omega_\theta)$$

Let $L_n \in \mathrm{Diff}_k(\overline{D})$ such that $L_n \to I$ when $n \to +\infty$. To prove that $j(L_n) \to j(I) = J(\Omega)$, we show that from any subsequence of L_n, we may extract a subsequence L_{i_n} such that $j(L_{i_n})$ converges to $j(I)$. The lemma 9 states that there is a subsequence L_{i_n} of bounded μ_k-variation that converges to I. Let $s \mapsto \theta_s$ be the associated mapping of the proposition 13. The proposition 14 yields the existence of of a $V \in \mathcal{V}_k(\overline{D})$ such that $\theta_s = T_s(V)$. By the directional continuity of J, we have $\lim_{s\to 0} J(\Omega_s(V)) = J(\Omega)$ and therefore

$$\lim_{n\to+\infty} j(L_{i_n}) = \lim_{n\to+\infty} J(\theta_{s_n}(\Omega)) = \lim_{n\to+\infty} J(\Omega_{s_n}(V)) = J(\Omega)$$

That concludes the proof. ■

2.5.5 Applications

In this article, we have studied the continuity issue by the way of continuous functionals. We provide in this section some equivalent formulations of the theorem 1 as well as some basic applications of our results.

Proposition 16 *Let $\mathcal{P} : \mathcal{A} \to \{\mathsf{true}, \mathsf{false}\}$ be a shape-dependent assertion. If*

$$\forall\, V \in \mathcal{V},\, \exists\, \tau \in \mathbb{R}_+^*,\, \forall\, s \in [0,\tau],\, \mathcal{P}(\Omega_s) = \mathsf{true} \tag{2.54}$$

there is a neighbourhood U of Ω such that

$$\mathcal{P} \equiv \mathsf{true} \text{ on } U$$

Proof – The property (2.54) states the directional continuity of $\mathcal{P} : \mathcal{A} \to X$ at Ω where $X = \{\mathsf{true}, \mathsf{false}\}$ is endowed with the discrete topology. As a consequence of the theorem 1, $\mathcal{P}$ is also (classicaly) continuous, and therefore, as $\{\mathsf{true}\}$ is a neighbourhood of true, there is a neighbourhood U of Ω such that for any $\omega \in U$, $\mathcal{P}(\omega) \in \{\mathsf{true}\}$, that is $\mathcal{P}(\omega) = \mathsf{true}$. ■

Proposition 17 *The set $A \subset \mathcal{A}$ is open if and only if*

$$\forall\, \Omega \in A,\, \forall\, V \in \mathcal{V},\, \exists\, \tau \in \mathbb{R}_+^*,\, \forall\, s \in [0,\tau],\, \Omega_s(V) \in A \tag{2.55}$$

Proof – If A is open, as for any $V \in \mathcal{V}$, $s \mapsto T_s(V) \in \mathcal{C}^1([0,1]; \mathcal{C}^k(\overline{D}; \mathbb{R}^n))$, for any $\Omega \in A$, there is a $\tau > 0$ such that $s \in [0, \tau]$ implies that $\Omega_s(V) = T_s(V)(\Omega) \in A$. Conversely, let us consider the shape-dependent property $\mathcal{P}$ defined by

$$\mathcal{P}(\Omega) = \begin{cases} \textsf{true} & \text{if } \Omega \in A \\ \textsf{false} & \text{otherwise.} \end{cases}$$

is directionally continuous at any $\Omega \in A$. Consequently it is continuous at any such Ω (lemma 16) and there is a neighbourhood U of Ω such that $\mathcal{P}(U) \subset \{\textsf{true}\}$, that is such that $U \subset A$. That concludes the proof. ■

Definition 7 *We say that the shape function $J : \mathcal{A} \to \mathbb{R}$ has a local directional minimum in Ω if*

$$\forall\, V \in \mathcal{V},\, \exists\, \tau \in \mathbb{R}_+^*,\, \forall\, s \in [0, \tau],\ J(\Omega) \leq J(\Omega_s) \tag{2.56}$$

Proposition 18 *Let $J : \mathcal{A} \to \mathbb{R}$ be a shape functional that has a local directional minimum in Ω. Then J has a local (classical) minimum in Ω.*

Proof – This is a simple consequence of the prop 16 when $\mathcal{P}$ is given by

$$\mathcal{P}(\omega) = \begin{cases} \textsf{true} & \text{if } J(\Omega) \leq J(\omega) \\ \textsf{false} & \text{otherwise.} \end{cases}$$

This shape function is directionaly continuous at Ω and therefore continuous at that point. On a suitable neighbourhood U of Ω, $\mathcal{P} \equiv \textsf{true}$, that is for $\omega \in U$, $J(\Omega) \leq J(\omega)$. ■

2.6 Appendix - Perturbation of the Identity. Topology

In the section 2.2, we have already defined the operator $\mathbf{\Omega}$ for any $\Omega \in \mathcal{A}$. We now define the subset of $\mathcal{A}$ constituted of perturbations of Ω by

$$\mathcal{A}[\Omega] = \mathbf{\Omega}(\mathrm{Diff}_k(\overline{D})) \tag{2.57}$$

The operator $\mathbf{\Omega}$ induces a topology on this set in the following way

$$\mathfrak{T}[\Omega] = \{\mathbf{\Omega}(\mathbf{O}), \mathbf{O} \text{ open set of } \mathrm{Diff}_k(\overline{D})\} \tag{2.58}$$

Proposition 19 $(\mathcal{A}[\Omega], \mathfrak{T}[\Omega])$ *is a topological space.*

Proof – (i) As $\varnothing = \mathbf{\Omega}(\varnothing)$ and $\mathcal{A}[\Omega] = \mathbf{\Omega}(\mathrm{Diff}_k(\overline{D}))$ both sets belong to $\mathfrak{T}[\Omega]$.

(ii) Let $\mathbf{\Omega}(\mathbf{O}_i)$, $i \in I$ be a family of elements of $\mathfrak{T}[\Omega]$. The set $\mathbf{O} = \cup_{i \in I} \mathbf{O}_i$ is open and

$$\cup_{i \in I} \mathbf{\Omega}(\mathbf{O}_i) = \mathbf{\Omega}(\cup_{i \in I} \mathbf{O}_i) = \mathbf{\Omega}(\mathbf{O})$$

Consequently, $\cup_{i \in I} \mathbf{\Omega}(\mathbf{O}_i)$ belongs to $\mathfrak{T}[\Omega]$.

(iii) Let $\mathbf{\Omega}(\mathbf{O}_i)$, $i \in I = \{1, ..., N\}$ be a finite sequence of sets of $\mathfrak{T}[\Omega]$. For any $\Omega' \in \cap_{i \in I} \mathbf{\Omega}(\mathbf{O}_i)$, there is a $(\theta_1, ..., \theta_N) \in \mathbf{O}_1 \times \cdots \times \mathbf{O}_N$ such that for any $i \in I$, $\mathbf{\Omega}(\theta_i) = \Omega'$.

For any $(i,j) \in I^2$, we define the mapping $\psi_i : \mathrm{Diff}_k(\overline{D}) \to \mathrm{Diff}_k(\overline{D})$ by

$$\psi_i(\theta) = \theta \circ [\theta_i^{-1} \circ \theta_1] \tag{2.59}$$

The mappings ψ_i are homeomorphisms of $\mathrm{Diff}_k(\overline{D})$ and $\psi_i(\theta_i) = \theta_1$.

We set $\mathbf{O}'_i = \psi_i(\mathbf{O}_i)$. The $\mathbf{O}'_i$ as well as $\mathbf{U}(\Omega') := \cap_{i\in I}\mathbf{O}'_i$ are open sets of $\mathrm{Diff}_k(\overline{D})$. For any $i \in I$, $\theta_i \in \mathbf{O}_i$ and therefore $\theta_1 \in \mathbf{O}'_i$. That implies that

$$\Omega' \in \mathbf{\Omega}(\mathbf{U}(\Omega')) \tag{2.60}$$

For any $\theta \in \mathbf{O}_i$, $\mathbf{\Omega}(\psi_i(\theta)) = \mathbf{\Omega}(\theta \circ [\theta_i^{-1} \circ \theta_1]) = \theta \circ [\theta_i^{-1} \circ \theta_1](\Omega)$ As every θ_i is one-to-one from Ω to Ω', $[\theta_i^{-1} \circ \theta_1](\Omega) = \Omega$. Therefore, $\mathbf{\Omega}(\psi_i(\theta)) = \theta(\Omega) = \mathbf{\Omega}(\theta)$, which implies $\mathbf{\Omega}(\mathbf{O}'_i) = \mathbf{\Omega}(\mathbf{O}_i)$. Finally, as $\mathbf{\Omega}(\cap_{i\in I}\mathbf{O}'_i) \subset \cap_{i\in I}\mathbf{\Omega}(\mathbf{O}'_i)$, we have

$$\mathbf{\Omega}(\mathbf{U}(\Omega')) \subset \cap_{i\in I}\mathbf{\Omega}(\mathbf{O}_i) \tag{2.61}$$

Thanks to (2.60) and (2.61), the set

$$\mathbf{U} = \cup_{\Omega'}\mathbf{U}(\Omega'),\ \Omega' \in \cap_{i\in I}\mathbf{\Omega}(\mathbf{O}_i) \tag{2.62}$$

is open and $\mathbf{\Omega}(\mathbf{U}) = \cap_{i\in I}\mathbf{\Omega}(\mathbf{O}_i)$. Therefore, $\cap_{i\in I}\mathbf{\Omega}(\mathbf{O}_i) \in \mathfrak{T}[\Omega]$. ■

Proposition 20 *Let Ω_1, $\Omega_2 \in \mathcal{A}$. We have $\mathcal{A}[\Omega_1] = \mathcal{A}[\Omega_2]$ if and only if there is a $\theta \in \mathrm{Diff}_k(\overline{D})$ such that $\Omega_2 = \theta(\Omega_1)$. In that case,*

$$(\mathcal{A}[\Omega_1], \mathfrak{T}[\Omega_1]) = (\mathcal{A}[\Omega_2], \mathfrak{T}[\Omega_2]) \tag{2.63}$$

Proof– If $\mathcal{A}[\Omega_1] = \mathcal{A}[\Omega_2]$, $\Omega_2 = \mathbf{\Omega_2}(I) \in \mathcal{A}[\Omega_2]$ and therefore $\Omega_2 \in \mathcal{A}[\Omega_1]$: there is a $\theta \in \mathrm{Diff}_k(\overline{D})$ such that $\Omega_2 = \theta(\Omega_1)$. Conversely, if there is a $\theta \in \mathrm{Diff}_k(\overline{D})$ such that $\Omega_2 = \theta(\Omega_1)$, then for any $\Omega \in \mathcal{A}[\Omega_2]$ given by $\Omega = \mathbf{\Omega_2}(\theta')$, we have $\Omega = \mathbf{\Omega_1}(\theta' \circ \theta)$. For any $\Omega \in \mathcal{A}[\Omega_1]$ given by $\Omega = \mathbf{\Omega_1}(\theta')$, we also have $\Omega = \mathbf{\Omega_2}(\theta' \circ \theta^{-1})$. Therefore, $\mathcal{A}[\Omega_1] = \mathcal{A}[\Omega_2]$.

Moreover, when $\mathbf{O}_2$ is an open set of $\mathrm{Diff}_k(\overline{D})$, $\mathbf{O}_1 := \{\theta' \circ \theta,\ \theta' \in \mathbf{O}_2\}$ is also open and that correspondance is one-to-one. As $\mathbf{\Omega_2}(\mathbf{O_2}) = \mathbf{\Omega_1}(\mathbf{O_1})$, we obtain $\mathfrak{T}[\Omega_1] = \mathfrak{T}[\Omega_2]$. ■

Definition 8 *Let Ω_i, $i \in I$ be a family of elements of $\mathcal{A}$ such that*

$$\mathcal{A} = \cup_{i\in I}\mathcal{A}[\Omega_i] \tag{2.64}$$

The perturbation of the identity topology $\mathfrak{T}$ *on $\mathcal{A}$ is the smallest topology such that $\mathfrak{T}_b \subset \mathfrak{T}$ with*

$$\mathfrak{T}_b = \cup_{i\in I}\mathfrak{T}[\Omega_i] \tag{2.65}$$

Thanks to the proposition 20, the definition of $\mathfrak{T}$ is independant of the choice of the Ω_i.

Régularité générique par rapport à la géométrie des équations quasilinéaires

Dans le chapitre précédent, on a mis en évidence le caractère fondamental de la non-singularité de l'équation de Navier-Stokes dans la démonstration de la régularité de l'état (u, p) par rapport à la géométrie. Lorsque la viscosité est faible cette propriété n'est pas nécessairement satisfaite. Pourtant, de nombreuses applications sont précisement des systèmes de faible viscosité. On peut alors se poser la question suivante : si l'équation linéarisé n'est pas systématiquement bien posée, l'est-elle au moins sur un ensemble dense de domaines dans la topologie ad hoc ?

A notre connaissance, ce problème reste ouvert. Toutefois, dans certains cas, nous savons répondre à une telle question pour les équations quasilinéaires scalaires de la forme

$$\begin{aligned} \operatorname{div}(A\nabla u) + B \circ (I, u, \nabla u) = f \quad & \text{dans } \Omega \\ u = g \quad & \text{sur } \Gamma \end{aligned}$$

oú Ω est un domaine borné de $\mathbb{R}^n$ de frontière $\partial\Omega = \Gamma$ et A, B, f et g peuvent dépendre de Ω. Ces équations présentent de nombreuses similarités avec les équations de Navier-Stokes, ce qui motive leur étude.

Sensibilité par rapport à la géométrie. On réalise ici une étude analogue à celle du premier chapitre. Toutefois l'analyse est menée dans des espaces de Hölder, plus adaptés à ces équations. Dans ce nouveau cadre, on met comme précédemment en évidence le fait que régularité des données et la non-singularité de l'équation impliquent l'existence de la dérivée (matérielle) de forme $\dot{u}$ de l'état u.

Généricité de l'existence de $\dot{u}$. Afin de prouver l'existence générique de $\dot{u}$, on cherche désormais à montrer que la non-singularité de l'équation quasilinéaire est satisfaite pour un ouvert dense de l'espace $\mathcal{D}$ des ouverts bornés de classe $\mathcal{C}^{2,\alpha}$.

On utilise les techniques de transversalité existant pour les opérateurs de Fredholm nonlinéaires. Pour une transformation θ du domaine de référence Ω, la solution u de l'équation dans le domaine Ω_θ est caractérisée comme solution d'une équation de la forme

$$\mathcal{A}(u \circ \theta, \theta) = 0$$

l'opérateur $\mathcal{A}$ étant tel que l'inversibilité de $\partial_u\mathcal{A}$ équivaut à la non-singularité de l'équation quasilinéaire. En plus de diverses hypothèses techniques, pour montrer que cette inversibilité est générique, il suffit de montrer la surjectivité de la différentielle totale $\mathrm{d}\mathcal{A} = \partial_u\mathcal{A} \cdot \mathrm{d}u + \partial_\theta\mathcal{A} \cdot \mathrm{d}\theta$. On montre quel type d'hypothèse simple sur les données permet de s'assurer que cette propriété est satisfaite.

Chapter 3

Generic Shape Regularity of Quasilinear Equations

3.1 Quasilinear Equations: Introduction

3.1.1 Prerequisite/ Technical Facts

3.1.1.1 Hölder mappings.

Definition as a global space. For any subset A of $\mathbb{R}^n$ and $\alpha \in (0,1]$, we denote $\mathcal{C}^{0,\alpha}(A;\mathbb{R}^m)$ the set of bounded mappings $f : A \to \mathbb{R}^m$ that satisfy the following Hölder condition of exponent α

$$H_\alpha(f) := \min\{K \in \mathbb{R}_+,\, \forall (x,y) \in A^2,\, |f(x) - f(y)| \leq K|x-y|^\alpha\} < +\infty \tag{3.1}$$

The norm on $\mathcal{C}^{0,\alpha}(A;\mathbb{R}^m)$ is defined by

$$\|f\|_{\mathcal{C}^{0,\alpha}} := \sup_{x\in A} |f(x)| + H_\alpha(f) \tag{3.2}$$

If U is an open subset of $\mathbb{R}^n$, for any $k \geq 0$, we define the functional space $\mathcal{C}^{k,\alpha}(\overline{U};\mathbb{R}^m)$ as the subspace of $\mathcal{C}^k(\overline{U};\mathbb{R}^m)$ whose functions satisfy

$$\|f\|_{\mathcal{C}^{k,\alpha}} = \sum_{|\gamma|<k} \|\partial^\gamma f\|_{\mathcal{C}^{0,\alpha}} < +\infty \tag{3.3}$$

with, as usual $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n$ and $\partial^\gamma f = \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n} f$.

Local characterization. Remember that the fact that a function is or not of class $\mathcal{C}^k$ can be characterized locally : the function $f : \overline{\Omega} \to \mathbb{R}$ belongs to $\mathcal{C}^k(\overline{\Omega};\mathbb{R})$ if and only if for any $x \in \overline{\Omega}$, there is a $r > 0$ such that the restriction of f to $\overline{\Omega} \cap \overline{B}(0,r)$ is $\mathcal{C}^k$.

The $\mathcal{C}^{k,\alpha}$ spaces are not defined locally so easily. However with a suitable regularity assumption on A, the local definition that we give in this section is proved to be equivalent to the global one.

For any $\rho > 0$ and $\alpha \in (0,1]$, we define the function

$$h_\alpha^\rho : x \mapsto H_\alpha(f_{A\cap B(x,\rho)}) \tag{3.4}$$

Definition 5 *Let A be a compact set of $\mathbb{R}^n$. We say that the mapping $f : A \to \mathbb{R}^m$ belong to $\widetilde{\mathcal{C}}^{0,\alpha}(A;\mathbb{R}^m)$ for a $\alpha \in (0;1]$ if there is a $\rho > 0$ such that*

$$H_\alpha^\rho(f) = \sup_{x\in A} h_\alpha^\rho(x) < +\infty \tag{3.5}$$

Conventionally, we set $H_\alpha^\infty(f) = H_\alpha(f)$.

The first step to compare the spaces $\widetilde{\mathcal{C}}^{0,\alpha}(A;\mathbb{R}^m)$ and $\mathcal{C}^{0,\alpha}(A;\mathbb{R}^m)$ lies in the

Lemma 19 *Assume that $f \in \widetilde{\mathcal{C}}^{0,\alpha}(A;\mathbb{R}^m)$. Then, the inequality (3.5) is satisfied for any $\rho \in (0;+\infty]$.*

Proof – Obviously, the mapping $\rho \in (0,+\infty] \mapsto h_\alpha^\rho(x) \in \mathbb{R}\cup\{+\infty\}$ is nondecreasing for any $x \in A$. Therefore, if the inequality (3.5) is satisfied with $\rho = \rho_0$, is also holds for any smaller value of ρ. For a $\rho > \rho_0$, we have

$$H_\alpha^\rho(f) \leq \max\left[H_\alpha^{\rho_0}(f)\,,\ \sup_{(x,y)\in K} \frac{|f(x)-f(y)|}{|x-y|^\alpha}\right] \tag{3.6}$$

where K is the compact set $K = (A\times A) - \{(x,y) \in A^2,\ \|x-y\| < \rho_0\}$. The condition (3.5) yields that $f \in \mathcal{C}^0(A;\mathbb{R})$. Therefore, the function $(x,y) \in K \mapsto |f(x)-f(y)|/|x-y|^\alpha$ is continuous, it is bounded, as well as $H_\alpha^\rho(f)$ by (3.6). ■

Corollary 3 *For any compact subset A of $\mathbb{R}^n$, the spaces $\widetilde{\mathcal{C}}^{0,\alpha}(A;\mathbb{R})$ and $\mathcal{C}^{0,\alpha}(A;\mathbb{R})$ are equal.*

Proof – The inclusion $\mathcal{C}^{0,\alpha}(A;\mathbb{R}) \subset \widetilde{\mathcal{C}}^{0,\alpha}(A;\mathbb{R})$ is clear. To prove the converse, we simply chose a $\rho > 0$ such that for any $x \in A$, $A \subset B(x,\rho)$. If the mapping f belongs to $\widetilde{\mathcal{C}}^{0,\alpha}(A;\mathbb{R})$ then we have the boundedness of h_α^ρ. As with that choice of ρ, h_α^ρ is constant and equal to $H_\alpha(f)$, f belongs to $\mathcal{C}^{0,\alpha}(A;\mathbb{R})$. ■

Proposition 21 *Let A be a compact and connected subset of $\mathbb{R}^n$ of class $\mathcal{C}^{2,\alpha}$. The semi-norms H_α^ρ of $\mathcal{C}^{0,\alpha}(A;\mathbb{R}^m)$ are equivalent for any $\rho \in (0;+\infty]$.*

Proof – Obviously for any $\rho \in (0;+\infty]$ and $f \in \mathcal{C}^{0,\alpha}(A;\mathbb{R}^m)$, we have $H_\alpha^\rho(f) \leq H_\alpha(f)$. The second inequality we are searching for is a consequence of the following geometrical property of A: there is an uniformly equicontinuous family $(\gamma_{x,y})_{(x,y)\in A^2}$ of paths indexed on $[0,1]$ connecting x and y. Consequently, there is a $N \in \mathbb{N}$ such that for any $(x,y) \in \mathcal{A}^2$, there is a sequence $(x_n)_{n\in\{1,\dots,N\}}$ of elements of A such that $x_1 = x$, $x_N = y$ and for any $n \in \{1,...,N-1\}$, $|x_{n+1}-x_n| < \rho$. For such a sequence, we have

$$\begin{aligned} |f(x)-f(y)| &\leq \sum_{n=1}^{N} |f(x_{n+1})-f(x_n)| \\ &\leq H_\alpha^\rho(f)\cdot\sum_{n=1}^{N-1}|x_{n+1}-x_n|^\alpha \leq \lambda_\alpha^N H_\alpha^\rho(f)|x-y|^\alpha \end{aligned}$$

for a $\lambda_\alpha^N > 0$ (see equation (3.24)) and therefore, $H_\alpha(f) \leq \lambda_\alpha^N H_\alpha^\rho(f)$. Therefore, for every $\rho \in (0;+\infty]$, H_α^ρ is equivalent to H_α and therefore they are all equivalent. ■

Lemma 20 *Let A be a compact an connected subset of $\mathbb{R}^n$ of class $\mathcal{C}^{2,\alpha}$. The space $\mathcal{C}^1(A;\mathbb{R}^m)$ is continuously embeded in $\mathcal{C}^{0,1}(A;\mathbb{R}^m)$*

Proof – Let $r>0$ be such that $A\subset B(0,r)$ and E be an extension operator:

$$E: \begin{array}{ccc} \mathcal{C}^1(A;\mathbb{R}^m) & \to & \mathcal{C}^1(\overline{B}(0,r),\mathbb{R}^m) \\ f & \mapsto & F=E(f) \end{array}$$

We have $F|_A = f$ and $\|F\|_{C^1}\le \|E\|\|f\|_{C^1}$. For any $(x,y)\in A^2$, we have

$$\begin{aligned} |f(x)-f(y)| = |F(x)-F(y)| &\le \int_0^1 |DF((1-t)x+ty)\cdot(y-x)|\,dt \quad (3.7)\\ &\le \|E\|\|f\|_{C^1}|x-y| \quad (3.8) \end{aligned}$$

That proves the lemma. ■

3.1.1.2 Smoothness of the boundary

We assume that the open set Ω where the quasilinear equation is solved is such that

(A1) The set Ω is a $\mathcal{C}^{2,\alpha}$-compact manifold of $\mathbb{R}^n$.

This assumption is useful to obtain the maximum regularity for the elliptic operators (see next section) and also to compare the the $\mathcal{C}^{0,1}$ and $\mathcal{C}^{1,\alpha}$ spaces.

3.1.1.3 The elliptic operator A

We assume that the operator $\mathrm{div}(A\nabla\cdot)$ is uniformly elliptic in $\overline{\Omega}$ and characterize the Hölder regularity of A

(A2) The mapping $x\mapsto A(x)$ belongs to $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})$ and there exist some constants $\nu_->0$ and $\nu_+>0$ such that

$$\forall x\in\overline{\Omega},\ \forall\xi\in\mathbb{R}^n,\ \ \nu_-|\xi|^2 \le^* \xi\cdot A(x)\cdot\xi \le \nu_+|\xi|^2 \quad (3.9)$$

Proposition 22 (Agmon-Douglis-Nirenberg) *Assume that (A1) and (A2) are satisfied. Then, the operator*

$$[\mathrm{div}(A\nabla\cdot)]^{-1}:\ \mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R})\times\mathcal{C}^{2,\alpha}(\Gamma;\mathbb{R})\to\mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}) \quad (3.10)$$

that associates to (f,g) the unique solution u of

$$\begin{array}{ll} \mathrm{div}(A\nabla u)=f & \text{in }\ \Omega \\ u=g & \text{on }\ \partial\Omega \end{array} \quad (3.11)$$

is an isomorphism. There is a $K>0$ that depends only on Ω, ν and a bound on $\|A\|_{\mathcal{C}^{1,\alpha}}$ such that for any (f,g), the solution u satisfies

$$\|u\|_{\mathcal{C}^{2,\alpha}}\le K[\|f\|_{\mathcal{C}^{0,\alpha}}+\|g\|_{\mathcal{C}^{2,\alpha}}] \quad (3.12)$$

3.1.1.4 Properties of the nonlinearity B

From now on, we assume that the mapping $(x,u,p)\mapsto B(x,u,p)$ satisfies the following assumption

(A3) For any bounded subset K of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$, B, $\partial_u B$ and $\partial_p B$ belong to $\mathcal{C}^{0,\alpha}(K)$. There exist $\alpha > 0$, $\beta \geq 0$, and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $x \in \overline{\Omega}$, $u \in \mathbb{R}$ and $M \in \mathbb{R}_+$

$$B(x, u, 0) \cdot u \leq -\alpha u^2 + \beta \tag{3.13}$$

$$|u| \leq M \text{ implies } |B(x, u, p)| \leq \gamma(M)(1 + |p|^2) \tag{3.14}$$

Remark 6 The set of assumptions (A1), (A2) and (A3) is classicaly required to to ensure the existence of a classical solution of the quasilinear equations, that is mappings u belonging to $\mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$, (see for example [32] th. 8.3 p. 301).

Proposition 23 *Let the assumptions* (A1) *and* (A3) *be satisfied. Then*
(i) *for any* $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$, $B \circ (I, u, \nabla u)$ *belongs to* $\mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R})$
(ii) *the mapping* $F : \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}) \to \mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R})$ *defined by*

$$F(u) = B \circ (I, u, \nabla u) \tag{3.15}$$

is continuously differentiable.
(iii) *the differential of* F *is given by*

$$DF(u) \cdot v = \partial_u B \circ (I, u, \nabla u) \cdot v + \partial_p B \circ (I, u, \nabla u) \cdot \nabla v \tag{3.16}$$

(iv) *for any* $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$, $DF(u)$ *is a compact operator.*

3.1.1.4.1 Proposition 23: proof of the point (i). We begin by the statement of two easy lemma on the Hölder mappings.

Lemma 21 ***Product of Hölder mappings.*** *Let* A *be a bounded subset of* $\mathbb{R}^n$ *and* f, g *be two mappings that belong to* $\mathcal{C}^{0,\alpha}(A; \mathbb{R})$. *Then* $fg \in \mathcal{C}^{0,\alpha}(A; \mathbb{R})$. *Moreover the product is a bilinear continuous operator from* $\mathcal{C}^{0,\alpha}(A; \mathbb{R}) \times \mathcal{C}^{0,\alpha}(A; \mathbb{R})$ *into* $\mathcal{C}^{0,\alpha}(A; \mathbb{R})$.

Proof – The mappings f and g are bounded on A. Precisely

$$\sup_{x \in A} |f(x)| \leq \|f\|_{\mathcal{C}^{0,\alpha}} \text{ and } \sup_{y \in A} |g(x)| \leq \|g\|_{\mathcal{C}^{0,\alpha}}$$

Therefore for any $(x, y) \in A^2$, we have the inequalities

$$\sup_{x \in A} |fg(x)| \leq \|f\|_{\mathcal{C}^{0,\alpha}} \|g\|_{\mathcal{C}^{0,\alpha}}$$

and

$$\begin{aligned} |fg(x) - fg(y)| &\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\leq 2\|f\|_{\mathcal{C}^{0,\alpha}} \|g\|_{\mathcal{C}^{0,\alpha}} |x - y|^\alpha \end{aligned}$$

That concludes the proof : fg belongs to $\mathcal{C}^{0,\alpha}(A; \mathbb{R})$ and the product is bilinear continous on this space. ■

Lemma 22 ***Compositions of Hölder mappings.*** *Let* $A \subset \mathbb{R}^n$ *and* $B \subset \mathbb{R}^m$. *Let* $f : B \to \mathbb{R}^p$, $g : A \to \mathbb{R}^m$ *be such that* $g(A) \subset B$. *Then, for any* $(\alpha, \beta) \in (0, 1]^2$
(i) *if* $f \in \mathcal{C}^{0,\alpha}(B; \mathbb{R}^p)$ *and* $g \in \mathcal{C}^{0,\beta}(A; \mathbb{R}^m)$, *then* $f \circ g \in \mathcal{C}^{0,\alpha\beta}(A; \mathbb{R}^p)$. *Moreover,*

$$H_{\alpha\beta}(f \circ g) \leq H_\alpha(f) \cdot [H_\beta(g)]^\alpha \tag{3.17}$$

(ii) *if for any bounded subset* K *of* B, $f|_K \in \mathcal{C}^{0,\alpha}(K; \mathbb{R}^p)$, A *is bounded and* $g \in \mathcal{C}^{0,\beta}(A; \mathbb{R}^m)$, *then* $f \circ g \in \mathcal{C}^{0,\alpha\beta}(A; \mathbb{R}^p)$.

Proof – Proof of (i): the boundedness of f yields the boundedness of $f \circ g$. Let K and L be some constants such that

$$\forall\,(x,y) \in B^2, x \neq y,\ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq K\ ,\ \forall\,(x,y) \in A^2, x \neq y,\ \frac{|g(x) - g(y)|}{|x-y|^\beta} \leq L$$

Then, for any $(x,y) \in A^2$, $x \neq y$, we have

$$\frac{|f \circ g(x) - f \circ g(y)|}{|x-y|^{\alpha\beta}} \leq K \frac{|g(x) - g(y)|^\alpha}{|x-y|^{\alpha\beta}} \leq KL^\alpha \tag{3.18}$$

The infimum on the admissible constants K and L yields (3.17).

Proof of (ii): g being Hölder continuous on the bounded set A, it is bounded. Let K be a bounded subset of B such that $g(A) \subset K$. Then $f|_K$ and g satisfy the assumptions of the point (i) and therefore, the same conclusion holds. ■

Remark 7 Notice that the property assumed in the point (ii) of the lemma 22 that

$$\text{for any bounded subset } K \text{ of } B,\ f|_K \in \mathcal{C}^{0,\alpha}(K;\mathbb{R}^p)$$

is *in general* stronger than the property $f \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(K;\mathbb{R}^p)$. It is equivalent for example if B is convex.

We finally prove the point (i) of the proposition 23. For any $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}))$, the mapping $x \mapsto (x, u(x), \nabla u(x))$ belongs to $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{2n+1})$ and therefore is Lipschitz on $\overline{\Omega}$.As $B \in \mathcal{C}^{0,\alpha}(K;\mathbb{R})$ for any bounded set $K \subset \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$, the point (ii) in the lemma 22 implies that $B \circ (I, u, \nabla u)$ belongs to $\mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R})$.

3.1.1.4.2 Proposition 23: proof of the points (ii) and (iii). First we notice that for any $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$, the mappings $\partial_u B \circ (I, u, \nabla u)$ and $\partial_p B \circ (I, u, \nabla u)$ belong to $\mathcal{C}^{0,\alpha}$: the proof is the same as in the point (i). Then, from the lemma 21 we deduce that $v \mapsto \partial_u B \circ (I, u, \nabla u) \cdot v + \partial_p B \circ (I, u, \nabla u) \cdot \nabla v$ is linear continuous from $\mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$ to $\mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R})$.

We set

$$f = B \text{ and } g = (u, \nabla u) \tag{3.19}$$

Obviously we have $\partial_2 f \circ (I, g) \cdot h = \partial_u B \circ (I, u, \nabla u) \cdot v + \partial_p B \circ (I, u, \nabla u) \cdot \nabla v$. To prove the points (ii) and (iii) it is therefore sufficient to show that

Proposition 24 *The remainder R_h given by*

$$R_h = f \circ (I, g + h) - f \circ (I, g) - \partial_2 f \circ (I, g) \cdot h \tag{3.20}$$

satisfies for any $g \in \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{n+1})$

$$R_h/\|h\|_{\mathcal{C}^{1,\alpha}} \to 0 \ \text{ in } \ \mathcal{C}^{0,\alpha} \ \text{ when } \ h \to 0 \ \text{ in } \ \mathcal{C}^{1,\alpha} \tag{3.21}$$

To study the remainder R_h, we introduce for any $x \in \overline{\Omega}$ and a, b in $\mathbb{R}^{n+1}$ the vector

$$r(x,a,b) = \int_0^1 [\partial_2 f(x,a) - \partial_2 f(x, a + tb)]\, dt \tag{3.22}$$

then we state and prove the following lemma

Lemma 23 *Let K be a bounded subset of $\overline{\Omega} \times \mathbb{R}^{n+1}$. Then*
(i) *For any $b \in \mathbb{R}^{n+1}$, the mapping*

$$r(\cdot,\cdot,b) : (x,a) \to r(x,a,b)$$

belongs to $\mathcal{C}^{0,\alpha}(K;\mathbb{R}^{n+1})$. Moreover, $r(\cdot,\cdot,b) \to 0$ in $\mathcal{C}^{0,\alpha}(K;\mathbb{R}^{n+1})$ when $b \to 0$.
(ii) *For any $(x,a) \in K$ and any bounded subset B of $\mathbb{R}^{n+1}$, the mapping*

$$r_B(x,a,\cdot) : b \in B \to r(x,a,b)$$

satsifies the Hölder condition with exponent α. Moreover, for any such B, there exist a uniform bound A_B such that

$$\forall\,(x,a) \in K,\ H_\alpha(r_B(x,a,\cdot)) < A_B$$

Proof – By assumption $\partial_u B$ and $\partial_p B$ (and therefore $\partial_2 f$) are $\mathcal{C}^{0,\alpha}$ on any bounded subset of $\overline{\Omega} \times \mathbb{R}^{n+1}$. Clearly, for any $b \in \mathbb{R}^{n+1}$, the set

$$K_b = \{(x, a+tb),\ (x,a) \in K,\ t \in [0,1]\} \tag{3.23}$$

is bounded and therefore, there exists a $L > 0$ such that for any (x,a) and (x',a') in K([1])

$$\begin{aligned}|r(x,a,b) - r(x',a',b)| &\leq \int_0^1 |\partial_2 f(x,a) - \partial_2 f(x',a')|\,dt \\ &\quad + \int_0^1 |\partial_2 f(x,a+tb) - \partial_2 f(x',a'+tb)|\,dt \\ &\leq 2L\,(|x-x'|^\alpha + |a-a'|^\alpha)\end{aligned}$$

and that proves the first part of the point (i). Clearly, $r(\cdot,\cdot,b) \to 0$ uniformly when $b \to 0$. To show that the convergence also occurs in $\mathcal{C}^{0,\alpha}$, we use a density argument. Let B be a bounded set of $\mathbb{R}^{n+1}$. We define the set

$$K(B) = \bigcup_{b \in B} K_b \tag{3.25}$$

(see the definition (3.23)) and we introduce for any $b \in B$ the linear operator $A_b : \mathcal{C}^{0,\alpha}(K(B)) \mapsto \mathcal{C}^{0,\alpha}(K)$ defined by

$$A_b : g \mapsto \left[(x,a) \mapsto \int_0^1 [g(x,a) - g(x,a+tb)]\,dt\right]$$

[1]A simple study shows that for any $\alpha \in \mathbb{R}_+^*$, there are two positive constants λ_α^- and λ_α^+ such that the following inequalities hold:

$$\forall (a,b) \in \mathbb{R}_+,\ \lambda_\alpha^-(a^\alpha + b^\alpha) \leq (a+b)^\alpha \leq \lambda_\alpha^+(a^\alpha + b^\alpha) \tag{3.24}$$

This property in mind, we show easily that the Hölder condition with exponent α for a n-variable mapping $f : A \to \mathbb{R}$ is equivalent to the existence of a $K > 0$ such that

$$\forall\,(x,y) \in A^2,\ |f(x) - f(y)| \leq K \sum_{i=1}^n |x_i - y_i|^\alpha$$

We notice that for any $\mathcal{C}^1$ mapping $g : \overline{\Omega} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $A_b(g)$ is continuously differentiable and converges to 0 in $\mathcal{C}^1(K; \mathbb{R}^{n+1})$ when $b \to 0$. Moreover,

$$\sup_{b \in B} \|A_b\| < +\infty$$

Let g_n be a sequence of $\mathcal{C}^1$-mappings such that $g_n \to \partial_2 f$ in $\mathcal{C}^{0,\alpha}(K(B))$. Then, we have

$$\begin{aligned} \|r(\cdot,\cdot,b)\|_{\mathcal{C}^{0,\alpha}} &= \|A_b(\partial_2 f)\|_{\mathcal{C}^{0,\alpha}} \\ &\leq \sup_{b \in B} \|A_b\| \cdot \|\partial_2 f - g_n\|_{\mathcal{C}^{0,\alpha}} + \|A_b(g_n)\|_{\mathcal{C}^{0,\alpha}} \end{aligned}$$

and therefore $r(\cdot,\cdot,b) \to 0$ in $\mathcal{C}^{0,\alpha}$ when $b \to 0$. That concludes the proof of the point (i).

The proof of the point (ii) is simple and follows the same line that the one of the first assertion of (i): again, we notice that the set $K(B)$ (see (3.25)) is bounded and we use the Hölder property of $\partial_2 f$ on that set to conclude the proof. ■

Now, we turn to the proof of the proposition 24. First, we notice that the remainder R_h satisfies

$$R_h = r \circ (I, g, h) \cdot h$$

and therefore that for any x, y in $\overline{\Omega}$

$$\frac{|R_h(x) - R_h(y)|}{|x-y|^\alpha} \leq A_h(x,y) \cdot h(x) + |r(y, g(y), h(y))| \cdot \frac{|h(y) - h(x)|}{|x-y|^\alpha} \tag{3.26}$$

with

$$\begin{aligned} A_h(x,y) &= \frac{|r(x,g(x),h(x)) - r(y,g(y),h(y))|}{|x-y|^\alpha} \\ &\leq \frac{|r(x,g(x),h(x)) - r(y,g(y),h(x))|}{|x-y|^\alpha} + \frac{|r(y,g(y),h(x)) - r(y,g(y),h(y))|}{|x-y|^\alpha} \end{aligned} \tag{3.27}$$

For any $g \in \mathcal{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{n+1})$, the set $K = \{(x, g(x)),\, x \in \overline{\Omega}\}$ is a bounded subset of $\overline{\Omega} \times \mathbb{R}^{n+1}$. When $h \to 0$ in $\mathcal{C}^{1,\alpha}$, $h \to 0$ uniformly on $\overline{\Omega}$. Therefore, from the point (i) of the lemma 23, we deduce that

$$|r(x,g(x),h(x)) - r(y,g(y),h(x))| \leq L_h(|x-y|^\alpha + |g(x) - g(y)|^\alpha)$$

with $L_h \to 0$ when $h \to 0$ in $\mathcal{C}^{1,\alpha}$. Then, if K is the Lipschitz constant of g, we obtain

$$\sup_{x \neq y} \frac{|r(x,g(x),h(x)) - r(y,g(y),h(y))|}{|x-y|^\alpha} \leq L_h(1+K) \to 0 \;\text{ when } h \to 0 \tag{3.28}$$

On the other hand, when $\mathrm{Range}(h)$ belongs to the bounded set B the existence of bound $A_B > 0$ such that for any x, y in $\overline{\Omega}$,

$$\frac{r(y,g(y),h(x)) - r(y,g(y),h(y))}{|x-y|^\alpha} \leq A_B \cdot \frac{|h(x) - h(y)|}{|x-y|^\alpha} \tag{3.29}$$

is provided by the point (ii). With (3.28) and (3.29), we conclude that

$$\sup_{x \neq y} |A_h(x,y)| \to 0 \;\text{ when } h \to 0 \;\text{ in }\; \mathcal{C}^{1,\alpha} \tag{3.30}$$

As the point (i) of the lemma 23 also yields

$$\sup_y |r(y, g(y), h(y))| \to 0 \text{ when } h \to 0 \text{ in } \mathcal{C}^{1,\alpha} \tag{3.31}$$

we conclude from the equation (3.26) that, as desired,

$$\sup_{x \neq y} \frac{|R_h(x) - R_h(y)|}{|x-y|^\alpha} \to 0 \text{ when } h \to 0 \text{ in } \mathcal{C}^{1,\alpha}$$

3.1.1.4.3 Proposition 23: proof of the point (iv). From the previous section, we know that $\partial_u B \circ (I, u, \nabla u)$ and $\partial_p B \circ (I, u, \nabla u)$ belong to $\mathcal{C}^{0,\alpha}(\overline{\Omega})$. If we set

$$f = {}^*(\partial_u B, \partial_p B) \circ (I, u, \nabla u)\,, \;\; g = (v, \nabla v) \tag{3.32}$$

$A = \overline{\Omega}$ and $m = 2n$, we can see that the following lemma proves the point (iv) of the proposition 23.

Lemma 24 *For any $f \in \mathcal{C}^{0,\alpha}(A; \mathbb{R}^m)$, the mapping $g \in \mathcal{C}^{1,\alpha}(A; \mathbb{R}^m) \mapsto \langle f, g \rangle_{\mathbb{R}^m} \in \mathcal{C}^{0,\alpha}(A; \mathbb{R})$ is compact.*

Proof – The mapping we consider is the composition of

$$g \in \mathcal{C}^{1,\alpha}(A; \mathbb{R}^m) \mapsto g \in \mathcal{C}^{0,\alpha}(A; \mathbb{R}^m) \mapsto \langle f, g \rangle_{\mathbb{R}^m} \in \mathcal{C}^{0,\alpha}(A; \mathbb{R})$$

It is well-know that the first is compact and the second is linear continuous (see lemma 22). It yields the result. ■

3.1.2 Spaces of shapes and transformations

Admissible transformations and sets. The admissibles geometries belong to

$$\mathcal{D} = \text{the set of open, bounded, } \mathcal{C}^{2,\alpha} \text{ sets of } \mathbb{R}^n \tag{3.33}$$

Definition 6 *Let Ω be a $\mathcal{C}^{2,\alpha}$ open bounded set of $\mathbb{R}^n$. We define the set of admissible transformations by*

$$\mathrm{Diff}_{2,\alpha}(\overline{\Omega}) = \{\theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n),\, \theta \textit{ is into and } \forall\, x \in \overline{\Omega},\, D\theta(x) \textit{ is invertible}\} \tag{3.34}$$

These transformations are compatible with the set of $\mathcal{C}^{2,\alpha}$ geometries:

Proposition 25 *Let Ω be a $\mathcal{C}^{2,\alpha}$ open bounded set of $\mathbb{R}^n$ and $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$. We set*

$$\Omega_\theta = \theta(\Omega) \;\; \textit{and} \;\; \Gamma_\theta = \theta(\Gamma) \tag{3.35}$$

(i) *Ω_θ is an open bounded set of $\mathbb{R}^n$ of class $\mathcal{C}^{2,\alpha}$ and $\Gamma_\theta = \partial\Omega_\theta$.*
(iii) *$\theta : \overline{\Omega} \to \overline{\Omega_\theta}$ is a $\mathcal{C}^{2,\alpha}$-diffeomorphism.*

Topological structure of the transformations and sets. First, we show that any suitably small perturbation of the identity belongs to the set of admissible transformations. This property implies that this set is open in $\mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n)$

Proposition 26 *Let Ω be a $\mathcal{C}^{2,\alpha}$ open bounded set of $\mathbb{R}^n$. There is a $\varepsilon > 0$ such that $\theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ and $\|\theta - I\|_{\mathcal{C}^{2,\alpha}} < \varepsilon$ implies that $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$.*

Proof – We check that the assumptions of the proposition 25 are satisfied. The set of $n \times n$ invertible matrices being open, there is a $\eta > 0$ such that

$$A \in \mathbb{R}^{n\times n} \text{ and } \|A - I_n\|_{\mathbb{R}^{n\times n}} < \eta \tag{3.36}$$

implies that A is invertible. For any $\varepsilon \leq \eta$ and any mapping $\theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$ such that $\|\theta - I\|_{\mathcal{C}^{2,\alpha}} < \varepsilon$, the condition (3.36) is satisfied with $A = D\theta(x)$ for any $x \in \overline{\Omega}$ and therefore $D\theta(x)$ is invertible.

Let $r > 0$ be such that $\overline{\Omega} \subset B(0,r)$. We introduce a linear continuous extension operator $E : \mathcal{C}^2(\overline{\Omega};\mathbb{R}^n) \to \mathcal{C}^2(\overline{B(0,r)};\mathbb{R}^n)$. For any $\varepsilon \leq 1/(2\|E\|r)$ an any θ such that $\|\theta - I\|_{\mathcal{C}^{2,\alpha}} < \varepsilon$, we have

$$\sup_{x\in\overline{B(0,r)}} \|DE(\theta)(x) - I_n\| \leq \frac{1}{2r} \tag{3.37}$$

Therefore, for any x and y in $\overline{\Omega}$, as $[x,y] \subset B(0,r)$, we have, using the inequality (3.37)

$$\|\theta(x) - \theta(y)\| \geq \|x-y\| - \|(\theta - I)(x) - (I + \theta - I)(y)\| \geq \frac{1}{2}\|x-y\|$$

and $\theta(x) = \theta(y)$ implies $x = y$. ■

Corollary 4 *The set* $\mathrm{Diff}_{2,\alpha}(\overline{\Omega})$ *is open in* $\mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$

Proof – Let $\theta_0 \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$. For any $\theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$, we have

$$\|\theta\circ\theta_0^{-1} - I\|_{\mathcal{C}^{2,\alpha}} = \|(\theta-\theta_0)\theta_0^{-1}\|_{\mathcal{C}^{2,\alpha}} \leq K\|\theta-\theta_0\|_{\mathcal{C}^{2,\alpha}}\|\theta_0^{-1}\|_{\mathcal{C}^{2,\alpha}}$$

for a $K > 0$. Therefore for any $\varepsilon > 0$, $\|\theta - \theta_0\|_{\mathcal{C}^{2,\alpha}} \leq \eta := \varepsilon/(K\|\theta_0^{-1}\|_{\mathcal{C}^{2,\alpha}})^{-1}$ implies $\|\theta\circ\theta_0^{-1} - I\| \leq \varepsilon$. The proposition 26 yields that for small enough values of η, $\theta\circ\theta_0^{-1} \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega_{\theta_0}})$ and therefore $\theta = (\theta\circ\theta_0^{-1})\circ\theta_0 \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$. ■

The topology on $\mathcal{D}$ is uniquely determined by a basis of neighbourhoods at every Ω:

Definition 7 (Topology on $\mathcal{D}$) *Let $\Omega \in \mathcal{D}$ and ε as in the proposition 26. A basis of neighbourhoods of Ω is given by $(V_r)_{0<r<\varepsilon}$ where*

$$V_r = \{\Omega_\theta,\, \theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n),\, \|\theta - I\|_{\mathcal{C}^{2,\alpha}} < r\} \tag{3.38}$$

3.1.3 Shape Analysis of the Quasilinear Equations

We study the behavior with respect to the shape of a class of quasilinear equations of the form

$$\begin{aligned} \operatorname{div}(A(x)\nabla u) + B(x,u,\nabla u) &= f \quad &\text{in } \Omega \\ u &= g &\text{on } \Gamma \end{aligned} \tag{3.39}$$

In the most general case, the applications A and B as well as the data f and g depend on the shape Ω. For any admissible Ω, A_Ω is a mapping from $\overline{\Omega}$ to $\mathbb{R}^{n\times n}$, B_Ω from $\overline{\Omega}\times\mathbb{R}^{n+1}$ and $f_\Omega : \overline{\Omega} \to \mathbb{R}$, $g_\Gamma : \Gamma \to \mathbb{R}$.

When there is no ambiguity on the reference set Ω, we set for simplicity for $X = A, B, f, g$ or any subsequent shape-dependent data

$$X_\theta := X_{\theta(\Omega)} \tag{3.40}$$

3.1.3.1 Regularity with respect to the shape

Definition 8 *Let Ω be a open set of $\mathbb{R}^n$ of class $\mathcal{C}^{2,\alpha}$ and $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$. For any function $f : \overline{\Omega_\theta} \to \mathbb{R}^m$, resp. $f : \overline{\Omega} \to \mathbb{R}^m$, we set*

$$\psi_\theta(f) = f \circ \theta \ \text{ resp. } \ \phi_\theta(f) = f \circ \theta^{-1} \tag{3.41}$$

Proposition 27 *For any $k \in \{0, 1, 2\}$, the formulas (3.41) define the isomorphisms $\psi_\theta : \mathcal{C}^{k,\alpha}(\overline{\Omega_\theta}; \mathbb{R}^m) \to \mathcal{C}^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m)$ and $\phi_\theta : \mathcal{C}^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m) \to \mathcal{C}^{k,\alpha}(\overline{\Omega_\theta}; \mathbb{R}^m)$.*

Proof – Obviously, for any $f : \overline{\Omega_\theta} \to \mathbb{R}^m$ and any $g : \overline{\Omega} \to \mathbb{R}^m$,

$$\phi_\theta \circ \psi_\theta(f) = f \ \text{ and } \ \psi_\theta \circ \phi_\theta(g) = g$$

As a consequence of the proposition 25, $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$ implies $\theta^{-1} \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega_\theta})$. As $\phi_\theta = \psi_{\theta^{-1}}$, it is sufficient to study the operators ψ_θ. When f belongs to $\mathcal{C}^{0,\alpha}(\overline{\Omega_\theta}; \mathbb{R}^m)$, as $\theta \in \mathcal{C}^{0,1}(\overline{\Omega}; \mathbb{R}^n)$, we have

$$\sup_{x\in\overline{\Omega}} |f \circ \theta(x)| \leq \sup_{x\in\overline{\Omega_\theta}} |f(x)| \ \text{ and } \ H_\alpha(f \circ \theta) \ \leq \ H_\alpha(f) \cdot [H_1(\theta)]^\alpha$$

(see equation (3.17) in the lemma 22). Therefore ψ_θ is a well-defined linear operator from $\mathcal{C}^{k,\alpha}(\overline{\Omega_\theta}; \mathbb{R}^m)$ to $\mathcal{C}^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m)$ when $k = 0$. For $k = 1$ or $= 2$, the proof is similar and based on estimates on the Hölder constants of products and composition of mappings (see lemma 21 and 22). These estimates are applied to the formulas $D(f \circ \theta) = Df \circ \theta \cdot D\theta$ and $D^2(f_i \circ \theta) = {}^tD\theta \cdot D^2 f_i \circ \theta \cdot + \partial_j f_i \circ \theta \cdot D^2\theta_j$. ■

Definition 9 (Regularity with respect to the shape)

Let k and k' be some elements of $\{0, 1, 2\}$.

Functions: *Let f be a shape-dependent mapping such that for any $\Omega \in \mathcal{D}$, $f_\Omega \in \mathcal{C}^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m)$. We say that f is n-times* differentiable w.r. to the shape *(*resp. of class $\mathcal{C}^n$ w.r. to the shape*) if the mapping $\theta \mapsto \psi_\theta(f_\theta)$ is n-times differentiable (resp. of class $\mathcal{C}^n$). When the right-hand side exists, we call* material derivative of f at Ω *the mapping*

$$\dot{f}_\Omega := \partial_\theta \left[\psi_\theta(f_{\theta(\Omega)})\right] |_{\theta=I} \tag{3.42}$$

Operators: *Let L be a shape-dependent operator such that for any $\Omega \in \mathcal{D}$, L_Ω is a linear continous mapping from $\mathcal{C}^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m)$ to $\mathcal{C}^{k',\alpha}(\overline{\Omega}; \mathbb{R}^m)$. We say that L is* n-times differentiable w.r. to the shape *(*resp. of class $\mathcal{C}^n$ w.r. to the shape*) if the mapping $\theta \mapsto \psi_\theta \circ L_{\theta(\Omega)} \circ \phi_\theta$ is n-times differentiable (resp. of class $\mathcal{C}^n$). When the right-hand side exists, we call* material derivative of L at Ω *the mapping*

$$\dot{L}_\Omega := \partial_\theta \left[\psi_\theta \circ L_\theta \circ \phi_\theta\right] |_{\theta=I} \tag{3.43}$$

Remark 8 These definitions admit straightforward generalizations, for example to the shape-dependent mappings and operators defined on $\Gamma = \partial\Omega$. We will not detail such cases here even if we will use such definitions later.

Proposition 28 *Let f be a shape differentiable mapping with values in $\mathbb{R}^m$. It is $\mathcal{C}^1$ w.r. to the shape iff its material derivative is $\mathcal{C}^0$ w.r. to the shape. Moreover for any $\theta_0 \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ and any $V \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n)$*

$$\partial_\theta \left[f_\theta \circ \theta\right] |_{\theta_0} \cdot V = [\psi_{\theta_0} \circ \dot{f}_{\theta_0} \circ \phi_{\theta_0}](V) \tag{3.44}$$

Proof – Let Ω be an open bounded set of $\mathbb{R}^n$ of class $\mathcal{C}^{2,\alpha}$ and $\theta_0 \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$. Let us define the linear continuous operator $S_N : \mathcal{C}^{k,\alpha}(\overline{\Omega};\mathbb{R}^N) \to \mathcal{C}^{k,\alpha}(\overline{\Omega_{\theta_0}};\mathbb{R}^N)$ (for $k = 1$ or 2) by

$$S_N(f) = f \circ \theta_0^{-1} \;\text{ or }\; S_N^{-1}(f) = f \circ \theta_0 \tag{3.45}$$

Then we have

$$f_{\theta(\Omega)} \circ \theta = \left[f_{(\theta\circ\theta_0^{-1})\circ\theta_0(\Omega)} \circ (\theta \circ \theta_0^{-1}) \right] \circ \theta_0 = S_m^{-1}[f_{S_n(\theta)[\theta_0(\Omega)]} \circ S_n(\theta)]$$

that is

$$f_{\theta(\Omega)} \circ \theta = S_m^{-1} \circ \left[\delta \mapsto f_{\delta(\theta_0(\Omega))} \circ \delta \right] \circ S_n(\theta) \tag{3.46}$$

As the operators S_n and S_m^{-1} are linear continuous and invertible operators and as $S_n(\theta_0) = I$, $\theta \mapsto f_{\theta(\Omega)} \circ \theta$ is differentiable at $\theta = \theta_0$ iff $\delta \mapsto f_{\delta(\theta_0(\Omega))} \circ \delta$ is differentiable at $\delta = I$, i.e. f is shape differentiable at $\theta_0(\Omega)$. When one of this statement holds, the chain rule gives for any $V \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$

$$\partial_\theta \left[f_\theta \circ \theta \right]|_{\theta_0} \cdot V = [dS_m^{-1}] \cdot \dot{f}_{\theta_0} \cdot dS_n(V)$$

As $dS_n = \phi_{\theta_0}$ and $dS_m^{-1} = \psi_{\theta_0}$, we obtain the equation (3.44). The equivalence between the continuity of $\partial_\theta \left[f_\theta \circ \theta \right]$ and the continuity of $\dot{f}_\theta$ is a direct consequence of the definition 9 for the shape-dependent operators. ■

3.1.3.2 Derivatives of the transformation mapping

Definition 10 *For any $\theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega},\mathbb{R}^n)$, we set*

$$J_\theta = D\theta \;\text{ and }\; \gamma_\theta = \det(D\theta) \tag{3.47}$$

Lemma 25 *Let $f_0 \in \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R})$ such that $f_0 > 0$ on $\overline{\Omega}$. Then, there is a neighbourhood V in $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R})$ of f_0 such that*

(i) *for any $f \in V$, $f > 0$ in $\overline{\Omega}$.*

(ii) *the mapping $\mathrm{Inv} : V \mapsto \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R})$ given by*

$$\mathrm{Inv}(f) = \frac{1}{f} \tag{3.48}$$

is continuously differentiable and

$$D\mathrm{Inv}(f) \cdot g = -\frac{g}{f^2} \tag{3.49}$$

Proof – First we show that an adequate choice of V implies that Inv has its values in $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R})$. As f_0 is continuous on $\overline{\Omega}$, there is a $m > 0$ such that $f \geq 2m$ and a neighbourhood V of f_0 such that $f \in V$ implies $f \geq m$. For any such f, $1/f$ is of class $\mathcal{C}^1$ and therefore, $1/f$ is $\mathcal{C}^{0,\alpha}$. Then, $-f'/f^2$ is the product of three $\mathcal{C}^{0,\alpha}$ mappings ; it is therefore $\mathcal{C}^{0,\alpha}$ (see lemma 21).

The mapping Inv is differentiable. Indeed, we have

$$\delta(f,g) := \frac{1}{f+g} - \frac{1}{f} + \frac{g}{f^2} = \frac{g^2}{f^2(f+g)}$$

As the product is multilinear continuous in $\mathcal{C}^{1,\alpha}$, we have

$$\|\delta(f,g)\|_{\mathcal{C}^{1,\alpha}} \leq \left[K \left\| \frac{1}{f} \right\|^2_{\mathcal{C}^{1,\alpha}} \left\| \frac{1}{f+g} \right\|_{\mathcal{C}^{1,\alpha}} \right] \|g\|^2_{\mathcal{C}^{1,\alpha}}$$

For g small enough, $\|1/(f+g)\|_{\mathcal{C}^{1,\alpha}}$ is bounded and $\delta(f,g) = \mathsf{o}(\|g\|_{\mathcal{C}^{1,\alpha}})$. Therefore, $D\mathrm{Inv}(f)$ exists and $D\mathrm{Inv}(f) \cdot g = -g/f^2$. By continuity of Inv, $f \mapsto 1/f^2$ is continuous with values in $\mathcal{C}^{1,\alpha}$ and therefore, $D\mathrm{Inv}$ is continuous.■

Lemma 26 *The mappings from* $\mathrm{Diff}_{2,\alpha}(\overline{\Omega};\mathbb{R})$ *to* $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R})$ *or* $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{3\times 3})$ *given by*

$$\theta \mapsto \gamma_\theta,\ \theta \mapsto \gamma_\theta^{-1},\ \theta \mapsto J_\theta \ \textit{ and } \ \theta \mapsto J_\theta^{-1}$$

are continuously differentiable and

$$\partial_\theta J_\theta|_{\theta=I} \cdot V = DV, \quad \partial_\theta \gamma_\theta|_{\theta=I} \cdot V = \mathrm{div}\, V \tag{3.50}$$

$$\partial_\theta J_\theta^{-1}|_{\theta=I} \cdot V = -DV, \quad \partial_\theta \gamma_\theta^{-1}|_{\theta=I} \cdot V = -\mathrm{div}\, V \tag{3.51}$$

Proof – (i) The differentiation mapping

$$D : \theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^3) \mapsto J_\theta = D\theta \in \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{3\times 3})$$

is linear continuous and therefore continuously differentiable. Therefore, $\partial_\theta J_\theta|_{\theta=I} \cdot V = \partial_\theta[D\theta]|_{\theta=I} \cdot V = D[\partial_\theta \theta \cdot V] = DV$.

(ii) for any $A \in \mathbb{R}^{3\times 3}$,

$$\det A = \sum_{\sigma \in P_n} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{3} a_{i,\sigma(i)}$$

The mapping $A \in \mathbb{R}^{3\times 3} \mapsto \det A \in \mathbb{R}$ is differentiable and $\mathrm{d}\det(I) \cdot H = \mathrm{tr}(H)$. On the other hand, as the sum and the product are continuously differentiable in $\mathcal{C}^{1,\alpha}$, the mapping

$$\mathbf{det} : A(\cdot) \in \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{3\times 3}) \mapsto \det A(\cdot) \in \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R})$$

is continuously differentiable. Therefore

$$\mathrm{d}\mathbf{det}([x \mapsto I]) \cdot H(\cdot) = \mathrm{tr}(H(\cdot))$$

(see the footnote [2]) and as $\gamma_\theta = \det(J_\theta)$, we have $\partial_\theta \gamma_\theta|_{\theta=I} \cdot V = \mathrm{tr}(DV) = \mathrm{div}\, V$.

[2] Let E, F and G be three Banach spaces. We consider some Banach spaces $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ and $(\mathcal{G}, \|\cdot\|_{\mathcal{G}})$ that are subsets of of F^E and G^E respectively. We assume that

(i) for any $x \in E$ there is a $L_x > 0$ such that for any $f \in \mathcal{F}$ (resp. $h \in \mathcal{G}$), we have

$$\|f(x)\|_F \leq L_x \|f\|_{\mathcal{F}},\ (\text{resp. } \|h(x)\|_G \leq L_x \|h\|_{\mathcal{G}})$$

(ii) for any $\lambda \in F$ (resp. $\mu \in G$), the constant mapping $f_\lambda : x \in E \mapsto \lambda \in F$ (resp. $h_\mu : x \in E \mapsto \mu \in G$) belongs to $\mathcal{F}$ (resp. $\mathcal{G}$) and there is a $K > 0$ (resp. $L > 0$) such that

$$\|f_\lambda\|_{\mathcal{F}} \leq K \|\lambda\|_F,\ (\text{resp. } \|h_\mu\|_{\mathcal{G}} \leq L \|\mu\|_G)$$

Under these assumptions, for any $x \in E$, the mapping $\delta_x^F : f \in \mathcal{F} \mapsto f(x) \in F$ (resp. $\delta_x^G : h \in \mathcal{G} \mapsto f(x) \in G$) is linear continuous and onto. The following schemes define therefore uniquely the isomorphisms $i_x^F : \mathcal{F}/\{\mathrm{Ker}\, \delta_x^F\} \to F$ and $i_x^G : \mathcal{G}/\{\mathrm{Ker}\, \delta_x^G\} \to G$

For any $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$, $\gamma_\theta > 0$. Using the lemma 25, we obtain the continuous differentiability of $\theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^3) \mapsto \gamma_\theta^{-1} \in \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R})$. Finally, the Cramer formula for the inverse of J_θ yields the continuous differentiability of $\theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^3) \mapsto J_\theta^{-1} \in \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{3\times 3})$. As for any invertible matrix X, $\partial_X X^{-1}|_{X=I} \cdot H = -H$, we have $\partial_\theta \gamma_\theta^{-1}|_{\theta=I} \cdot V = -\partial_\theta \gamma_\theta|_{\theta=I} \cdot V = -\mathrm{div}\, V$ as well as $\partial_\theta J_\theta^{-1}|_{\theta=I} \cdot V = -\partial_\theta J_\theta|_{\theta=I} \cdot V = DV$ (see again the previous footnote). ■

3.1.3.3 Transported Quasilinear Operator. Definition, Explicit Expression and Regularity

Definition 11 *For any $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$ and $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$, we set*

$$E_1(\theta,u) = \psi_\theta \circ [\mathrm{div}(A_\theta \nabla \cdot) + B_\theta \circ (I, \cdot, \nabla \cdot)] \circ \phi_\theta(u) - \psi_\theta(f_\theta) \quad (3.56)$$
$$E_2(\theta,u) = u - \psi_\theta(g_\theta) \quad (3.57)$$

Proposition 29 (Main Property of $E = (E_1, E_2)$)
Let $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$ and $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$. We have $E(\theta,u) = 0$ if and only if

$$\begin{aligned} \mathrm{div}(A_\theta \nabla \phi_\theta(u)) + B_\theta \circ (I, \phi_\theta(u), \nabla\phi_\theta(u))) = f_\theta \quad & \text{in } \Omega_\theta \\ \phi_\theta(u) = g_\theta \quad & \text{on } \Gamma_\theta \end{aligned}$$

Proof – The proposition 27 has established that ψ_θ and ϕ_θ were two isomorphisms such that $\psi_\theta \circ \phi_\theta = I$ and $\phi_\theta \circ \psi_\theta = I$. Consequently, $E_1(\theta,u) = 0$ iff $\phi_\theta(E_1(\theta,u)) = \mathrm{div}(A_\theta \nabla \phi_\theta(u)) + B_\theta \circ (I, \phi_\theta(u), \nabla \phi_\theta(u))) - f_\theta = 0$. On the other hand, $E_2(\theta,u) = 0$ iff $\phi_\theta(E_2(\theta,u)) = \phi_\theta(u) - g_\theta = 0$ as desired. ■

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\delta_x^F"] \arrow[d, "\pi_x^F"'] & F \\ \mathcal{F}/\{\mathrm{Ker}\,\delta_x^F\} \arrow[ur, "i_x^F"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} \arrow[r, "\delta_x^G"] \arrow[d, "\pi_x^G"'] & G \\ \mathcal{G}/\{\mathrm{Ker}\,\delta_x^G\} \arrow[ur, "i_x^G"'] & \end{tikzcd} \quad (3.52)$$

Lemma 27 *We assume that E, F and G satisfy the previous assumptions. Let $g : F \to G$ be such that the mapping $\mathrm{g} : f \in \mathcal{F} \mapsto g \circ f \in \mathcal{G}$ is everywhere differentiable. Then, the mapping $g : F \mapsto G$ is everywhere differentiable and for any $x \in E$ and $h \in \mathcal{F}$*

$$[d\mathrm{g}(f) \cdot h](x) = d[g](f(x)) \cdot h(x) \quad (3.53)$$

Proof – As $f(x) = f'(x)$ implies $g \circ f(x) = g \circ f'(x)$, $f - f' \in \mathrm{Ker}\,\delta_x^F$ implies $\delta_x^G \circ \mathbf{g}(f) = \delta_x^G \circ \mathbf{g}(f')$ and the following scheme defines uniquely the mapping κ_x.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\delta_x^G \circ \mathbf{g}"] \arrow[d, "\pi_x^F"'] & G \\ \mathcal{F}/\{\mathrm{Ker}\,\delta_x^F\} \arrow[ur, "\kappa_x"'] & \end{tikzcd} \quad (3.54)$$

We define $\tilde{\kappa}_x = \kappa_x \circ (i_x^F)^{-1}$. Then, we have $\delta_x^G \circ \mathbf{g} = \kappa_x \circ \pi_x^F = \tilde{\kappa}_x \circ i_x^F \circ \pi_x^F = \tilde{\kappa}_x \circ \delta_x^F$ and therefore, as for any $f \in \mathcal{F}$, $g(f(x)) = \delta_x^G \circ \mathbf{g}(f)$ and $\tilde{\kappa}_x \circ \delta_x^F(f) = \tilde{\kappa}_x(f(x))$ we obtain $\tilde{\kappa}_x = g$ and finally

$$\delta_x^G \circ \mathbf{g} = g \circ \delta_x^F \quad (3.55)$$

The assumption (ii) yields the continuity of the linear operator $[y \mapsto f_y]$. Therefore, (3.55) gives $g = \delta_x^G \circ \mathbf{g} \circ [y \mapsto f_y]$. The differentiability of the r.h.s yields the differentiability of g.

The differentiation of (3.55) at f yields (3.53). ■

Proposition 30 (Explicit expressions) *For any $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$, we have*

$$\psi_\theta \circ [\mathrm{div}(A_\theta \nabla \cdot)] \circ \phi_\theta = \gamma_\theta^{-1} \mathrm{div}(A^\theta \nabla u) \text{ with } A^\theta = \gamma_\theta \, {}^* J_\theta^{-1} (A_\theta \circ \theta) J_\theta^{-1} \quad (3.58)$$

$$\psi_\theta \circ [B_\theta(I, \cdot, \nabla \cdot)] \circ \phi_\theta = B_\theta \circ (\theta, u, {}^* J_\theta^{-1} \cdot \nabla u) \quad (3.59)$$

$$\psi_\theta(f_\theta) = f_\theta \circ \theta \quad (3.60)$$

Proof – We only prove the equation (3.58), the two others being simpler. For any $\phi \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$, with $\mathrm{Supp}(\phi) \subset \Omega$, we have

$$\int_\Omega [\mathrm{div}\,(A_\theta \nabla (u \circ \theta^{-1})] \circ \theta \cdot \phi \, dx = \int_\Omega [\mathrm{div}\,(A_\theta \nabla (u \circ \theta^{-1})) \cdot (\gamma_\theta^{-1} \phi) \circ \theta^{-1}] \circ \theta \, \gamma_\theta dx$$

and therefore by the change of variable $y = \theta(x)$, we obtain

$$\begin{aligned}
\int_\Omega [\mathrm{div}\,(A_\theta \nabla (u \circ \theta^{-1})] \circ \theta \cdot \phi \, dx &= \int_{\Omega_\theta} [\mathrm{div}\,(A_\theta \nabla (u \circ \theta^{-1})) \cdot (\gamma_\theta^{-1} \phi) \circ \theta^{-1}] \, dy \\
&= \int_{\Omega_\theta} \left\langle A_\theta \nabla (u \circ \theta^{-1}), \nabla ((\gamma_\theta^{-1} \phi) \circ \theta^{-1}) \right\rangle \, dy
\end{aligned}$$

As $\nabla (f \circ \theta^{-1}) = {}^* J_\theta^{-1} \cdot (\nabla f) \circ \theta^{-1}$, we have

$$\begin{aligned}
&\int_\Omega [\mathrm{div}\,(A_\theta \nabla (u \circ \theta^{-1})] \circ \theta \cdot \phi \, dx = \\
&\qquad \int_{\Omega_\theta} \left\langle J_\theta^{-1} (A_\theta \circ \theta) \, {}^* J_\theta^{-1} \nabla u, \nabla (\gamma_\theta^{-1} \phi) \right\rangle \circ \theta^{-1} \, dy = \\
&\qquad \int_{\Omega_\theta} \left\langle A^\theta \nabla u, \nabla ((\gamma_\theta^{-1} \phi) \right\rangle \circ \theta^{-1} \, (\gamma_\theta^{-1} \circ \theta^{-1}) dy
\end{aligned}$$

and consequently, by the change of variables $x = \theta^{-1}(y)$

$$\begin{aligned}
\int_\Omega [\mathrm{div}\,(A_\theta \nabla (u \circ \theta^{-1})] \circ \theta \cdot \phi \, dx &= \int_\Omega \left\langle A^\theta \nabla u, \nabla ((\gamma_\theta^{-1} \phi) \right\rangle \, dx \\
&= \int_\Omega \gamma_\theta^{-1} \mathrm{div}(A^\theta \nabla u) \cdot \phi \, dx
\end{aligned}$$

That proves the equation (3.58). ■

In the following proposition, we make a sequence of regularity and structure assumptions on the shape-dependent data. These assumptions define some "dot" mappings that are not strictly speaking the material derivatives of the associated mapping but uniquely determine them. They can be considered as reduced material derivatives.

Proposition 31 *Assume that*

For any reference set Ω

- the mapping $\theta \mapsto A_\theta \in \mathcal{C}^{1,\alpha}(\overline{\Omega_\theta}; \mathbb{R}^{n\times n})$ is $\mathcal{C}^1$ w.r. to the shape and there is a mapping $\dot{A}$ in $\mathcal{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{n^3})$ such that for all $x \in \overline{\Omega}$

$$[\partial_\theta A_\theta|_{\theta=I} \cdot (\delta\theta)](x) = \dot{A}(x) \cdot (\delta\theta)(x) \tag{3.61}$$

- the mapping $(\theta, u, p) \to B_\theta \circ (\theta, u, p)$ is continuously differentiable from $\mathrm{Diff}_{2,\alpha}(\overline{\Omega}) \times \mathcal{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}) \times \mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ to $\mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R})$ and there is a mapping $\dot{B}$ in $\mathcal{C}^{0,\alpha}(K; \mathbb{R}^n)$ for any compact $K \subset \overline{\Omega} \times \mathbb{R}^{n+1}$ such that for all $x \in \overline{\Omega}$

$$[\partial_\theta B_\theta(\theta, u, p)|_{\theta=I} \cdot \delta\theta](x) = \langle \dot{B}(x, u(x), p(x)), (\delta\theta)(x)\rangle_{\mathbb{R}^n} \tag{3.62}$$

- the mappings $f : \theta \mapsto f_\theta \in \mathcal{C}^{0,\alpha}(\overline{\Omega_\theta}; \mathbb{R})$, $g : \theta \mapsto g_\theta \in \mathcal{C}^{2,\alpha}(\Gamma_\theta; \mathbb{R})$ are $\mathcal{C}^1$ w.r. to the shape and there is some $\dot{f} \in \mathcal{C}^{0,\alpha}(\overline{\Omega_\theta}; \mathbb{R}^n)$, $\dot{g} \in \mathcal{C}^{2,\alpha}(\Gamma_\theta; \mathbb{R}^n)$ such that for all $x \in \overline{\Omega}$, resp. for all $x \in \Gamma$

$$[\partial_\theta f_\theta|_{\theta=I} \cdot (\delta\theta)](x) = \langle \dot{f}(x), (\delta\theta)(x)\rangle_{\mathbb{R}^n} \tag{3.63}$$

$$\text{resp. } [\partial_\theta g_\theta|_{\theta=I} \cdot (\delta\theta)](x) = \langle \dot{g}(x), (\delta\theta)(x)\rangle_{\mathbb{R}^n} \tag{3.64}$$

Then, the mapping

$$E = (E_1, E_2) : \mathrm{Diff}_{2,\alpha}(\overline{\Omega}) \times \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}) \mapsto \mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Gamma; \mathbb{R}) \tag{3.65}$$

defined by the equations (3.56) and (3.57) is continuously differentiable.

Remark 9 Obviously, the definition of the mapping E is dependent of the reference set Ω in $\mathcal{D}$. This dependence is emphasized in the notation E^Ω when needed.

Proof –

(i) At first, we notice that the assumption on A_θ and the lemma 26 yield the continuous differentiability of $\theta \mapsto \gamma_\theta^{-1}$ and $\theta \mapsto A^\theta = \gamma_\theta\, {}^*J_\theta^{-1}(A_\theta \circ \theta)J_\theta^{-1}$ in $\mathcal{C}^{1,\alpha}$. Therefore, the mapping $F : (\theta, u) \mapsto \gamma_\theta^{-1}\mathrm{div}(A^\theta \nabla u)$ is continuously differentiable, with

$$\partial_u F(\theta, u) = \gamma_\theta^{-1}\mathrm{div}(A^\theta \nabla(\cdot))$$

and

$$\partial_\theta F(\theta, u) = \partial_\theta \gamma_\theta^{-1}\mathrm{div}(A^\theta \nabla u) + \gamma_\theta^{-1}\mathrm{div}(\partial_\theta A^\theta \nabla u)$$

(ii) The mapping $\psi_\theta \circ B_\theta \circ (I, \cdot, \nabla \cdot) \circ \phi_\theta$ is continuously differentiable: to see that, we first use the explicit expression given by the equation (3.59): $\psi_\theta \circ [B_\theta(I, \cdot, \nabla \cdot)] \circ \phi_\theta = B_\theta \circ (\theta, u, {}^*J_\theta^{-1} \cdot \nabla u)$. As the mapping $(\theta, u) \mapsto (\theta, u, {}^*J_\theta^{-1} \cdot \nabla u)$ is continuously differentiable (with values in $\mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n) \times \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}) \times \mathcal{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$), the assumption (ii) on B_θ yields the continuous differentiability of the composed mapping.

(iii) The right-hand side f_θ is constant with respect to u and the continuous differentiability of $f_\theta \circ \theta$ is true by assumption.

(iv) The regularity of $\theta \mapsto g_\theta \circ \theta$ is a direct consequence of the assumptions. It yields the continuous differentiability of E_2. ■

Regularity of the Linearized System. Shape differentiability of u

Definition 12 *The quasilinear system (3.39) is* regular *in Ω if for any solution u in Ω and any $h \in \mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R})$, $k \in \mathcal{C}^{2,\alpha}(\Gamma;\mathbb{R})$, the linearized system*

$$\begin{aligned} \operatorname{div}(A(x)\nabla y) + \partial_u B(x,u,\nabla u)\cdot y + \partial_p B(x,u,\nabla u)\cdot \nabla y = h \quad & \text{in } \Omega \\ y = k \quad & \text{on } \Gamma \end{aligned} \tag{3.66}$$

has a unique solution $y \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$.

Theorem 4 (Shape Regularity of the solutions) *Let the assumptions of the proposition 31 be satisfied. Let u_r be a solution of the system (3.39) in $\Omega \in \mathcal{D}$.*

Assume that the system (3.39) is regular in Ω.

There is a unique shape-dependent mapping u defined in a neighbourhood V_ρ of Ω such that

Regularity: *The mapping u is $\mathcal{C}^1$ with respect to the shape with values in $\mathcal{C}^{2,\alpha}$.*

Solution: *For any transformation θ such that $\|\theta - I\|_{\mathcal{C}^{2,\alpha}} < \rho$, u_θ is a solution of the system (3.39) in Ω_θ.*

Initial Condition: *In Ω, u is equal to u_r.*

Proof – For any $\theta \in \operatorname{Diff}_{2,\alpha}(\overline{\Omega})$, and $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$, $\phi_\theta(u)$ is a solution of the quasilinear system in Ω_θ if and only if $E(\theta,\phi_\theta(u)) = 0$. The statements of the theorem are provided by the implicit function theorem : $E(I,u_r) = 0$, E is a $\mathcal{C}^1$ mapping whose partial differential $\partial_u E$ is an isomorphism by assumption. That yields the local existence and regularity of $\theta \mapsto u_\theta \circ \theta^{-1}$. ■

3.2 Fredholm Operators

3.2.1 Definitions. Transversality Theorem

We give a short review of some definitions and results of the Fredholm Operators Theory. We refer the reader to the excellent [42] for further details on this field. Some of the properties that we need on this subject may also be found in [25] and [43].

Let E and F be two Banach spaces.

Definition 13 (Linear Fredholm Operators) *The operator $A \in \mathcal{L}(E,F)$ is* Fredholm *if $\operatorname{Ker}(A)$ and $\operatorname{Coker}(A) = F/A(E)$ have finite dimension. The* index *of A is defined by:*

$$\operatorname{ind}(A) = \dim \operatorname{Ker}(A) - \dim \operatorname{CoKer}(A) \tag{3.67}$$

Definition 14 (Nonlinear Fredholm Operators) *Let U be an open subset of E. The $\mathcal{C}^1$ mapping $\mathcal{A} : U \to F$ is* Fredholm *if for any $x \in E$, its differential $d\mathcal{A}(x)$ is Fredholm. In that case, the index of $d\mathcal{A}(x)$ is locally independent of x and we set*

$$\operatorname{ind}(\mathcal{A}) = \operatorname{ind}(d\mathcal{A}(x)) \tag{3.68}$$

Definition 15 *Let $\mathcal{A} : U \to F$ be a Fredholm mapping. The point $x \in E$ is* regular *if $d\mathcal{A}(x)$ is onto. Otherwise it is* singular*. The images of the singular points by $\mathcal{A}$ are the* singular values *of $\mathcal{A}$. Their complement in F are the* regular values *of $\mathcal{A}$.*

Proposition 32 *The sum of an isomorphism $A : E \to F$ and of a compact operator $B : E \to F$ is Fredholm of index 0.*

Theorem 5 (Transversality theorem). *Let M, E, F be three Banach spaces and $U \subset E$, $V \subset M$ two open sets. Let $\mathcal{A} : U \times V \to F$ be a $\mathcal{C}^k$ mapping and $y \in F$. We assume that:*

(i) Fredholm property: *For any $\mu \in V$, the mapping $\mathcal{A}_\mu : x \mapsto \mathcal{A}(x, \mu)$ is Fredholm and* $\mathrm{ind}(A_\mu) < k$.

(ii) Regularity: *The element y is regular value of $\mathcal{A}$.*

(iii) Properness: *For any compact set K of U, the set*

$$\{\mu \in V, \exists\, x \in K,\, \mathcal{A}(x, \mu) = y\}$$

is relatively compact.

Then, the set of $\mu \in V$ such that y is a regular value of $\mathcal{A}_\mu$ is a dense open subset of V.

let $\Omega \in \mathcal{D}$. We intend to apply the previous theorem with

$$V = \mathrm{Diff}_{2,\alpha}(\overline{\Omega}) \subset M = \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^3) \tag{3.69}$$

$$U = E = \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}) \text{ and } F = \mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Gamma; \mathbb{R}) \tag{3.70}$$

$$\mathcal{A} = E = (E_1, E_2) \text{ as defined in (3.56), (3.57)} \tag{3.71}$$

$$k = 1 \quad \text{and} \quad y = 0 \tag{3.72}$$

Some of the assumptions of the theorem 5 are easily checked : M is clearly a Banach space and the corollary 4 states that V is open. The mapping $E = (E_1, E_2)$ is also of class $\mathcal{C}^1$ (see the proposition 31).

The other assumptions of the theorem are checked in the following sections. They follow from the following set of assumptions on A_θ and B_θ. They are in the spirit of the classical assumptions (A1), (A2) and (A3) but they take into account the shape-dependent nature of the problem.

(A1) The admissible sets Ω are open and bounded subset of $\mathbb{R}^n$ of class $\mathcal{C}^{2,\alpha}$.

(A2) For any $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$, there exist $\nu_-(\theta) > 0$ and $\nu_+(\theta) > 0$ such that

$$\forall\, x \in \overline{\Omega_\theta},\ \forall\, \xi \in \mathbb{R}^n,\ \ \nu_-(\theta) \cdot |\xi|^2 \le {}^*\xi \cdot A_\theta(x) \cdot \xi \le \nu_+(\theta) \cdot |\xi|^2 \tag{3.73}$$

(A3) for any $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$, there is a neighbourhood V_θ of θ such that :

• for any open bounded subset U of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$, and $\theta' \in V_\theta$, $B_{\theta'} \circ (\theta'(\cdot), \cdot, \cdot)$ belongs to $\mathcal{C}^{1,\alpha}(\overline{U}; \mathbb{R})$ and this $\mathcal{C}^{1,\alpha}$-norm is uniformly bounded with respect to $\theta' \in V_\theta$. Moreover, $(\partial_x, \partial_u, \partial_p)\nabla_p B$ exists and belongs to $\mathcal{C}^{0,\alpha}(\overline{U}; \mathbb{R}^{(2n+1)\times n})$.

• there exist $\alpha(\theta) > 0$, $\beta(\theta) \ge 0$, and $\gamma(\theta) : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $\theta' \in V_\theta$, $x \in \overline{\Omega_\theta}$, $u \in \mathbb{R}$ and $M \in \mathbb{R}_+$

$$B_{\theta'}(x, u, 0) \cdot u \le -\alpha(\theta) \cdot u^2 + \beta(\theta) \tag{3.74}$$

and $|u| \le M$ implies

$$|B_{\theta'}(x, u, p)| \le \gamma(\theta)(M)(1 + |p|^2) \tag{3.75}$$

3.2.2 Assumption (i) : Fredholm Property

Proposition 33 *For any $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$, the mapping $u \mapsto E(\theta, u)$ is Fredholm of index 0.*

Proof – For any $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$, Ω_θ satisfies the assumption (A1), and A_θ satisfies (A2). Therefore, there is a unique solution u to the linear elliptic system.

$$\begin{aligned} \mathrm{div}(A_\theta \nabla u) = f_\theta \quad & \text{in } \Omega_\theta \\ u = g_\theta \quad & \text{on } \Gamma_\theta \end{aligned} \tag{3.76}$$

and moreover

$$\|u\|_{\mathcal{C}^{2,\alpha}} \leq K_\theta(\|f\|_{\mathcal{C}^{0,\alpha}} + \|g\|_{\mathcal{C}^{2,\alpha}}) \tag{3.77}$$

(see [32]). In other words, the mapping $(\mathrm{div}(A_\theta \nabla \cdot), \cdot|_{\Gamma_\theta})$ is a isomorphism. As ψ_θ and ϕ_θ are isomorphisms, $\mathbf{A}_\theta = \psi_\theta \circ (\mathrm{div}(A_\theta \nabla \cdot), \cdot|_{\Gamma_\theta}) \circ \phi_\theta$ is also an isomorphism. On the other hand the mapping

$$u \in \mathcal{C}^{2,\alpha}(\overline{\Omega_\theta}; \mathbb{R}) \mapsto (B_\theta \circ (I, u, \nabla u), 0) \in \mathcal{C}^{0,\alpha}(\overline{\Omega_\theta}; \mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Gamma_\theta; \mathbb{R})$$

is $\mathcal{C}^1$ and its differential is compact (see proposition 23). Consequently, $\mathbf{B}_\theta := \psi_\theta \circ (B_\theta \circ (I, \cdot, \nabla \cdot), 0) \circ \phi_\theta$ is also $\mathcal{C}^1$ and its differential is compact. As $E(\theta, \cdot) = \mathbf{A}_\theta + \mathbf{B}_\theta$, by the proposition 32, $E(\theta, \cdot)$ is Fredholm of index 0. ∎

3.2.3 Assumption (iii) : Properness

Our aim in this section is to prove the

Proposition 34 *For any compact set $K \subset \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$, the set*

$$A_K = \{u \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}),\ E(\theta, u) = 0 \ \text{ for a } \ \theta \in K\} \tag{3.78}$$

is relatively compact.

Proof – First, we notice that the set A_K is relatively compact if and only if for any sequences $u_n \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$ and $\theta_n \in K$ such that $E(\theta_n, u_n) = 0$, there is a converging subsequence of u_n.

By compactness of K, it is sufficient to show that for any converging subsequence $\theta_n \to \theta \in V$ and any $u_n \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$ such that $E(\theta_n, u_n) = 0$ there is a converging subsequence (still noted u_n) and a $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$ such that $u_n \to u$.

The existence of such a u is proved with the use of the estimates provided by the lemmas 29 and 30 whose proof is postponed. We notice that the solutions u_n of $E(\theta_n, u_n) = 0$ satisfy

$$\begin{aligned} \mathrm{div}(A^{\theta_n} \nabla u) + B_{\theta_n} \circ (\theta_n, u, {}^*J_{\theta_n}^{-1} \nabla u) = f_{\theta_n} \circ \theta_n \quad & \text{in } \Omega \\ u = g_{\theta_n} \circ \theta_n \quad & \text{on } \Gamma \end{aligned}$$

We write that system under the form

$$\begin{aligned} \mathrm{div}(\mathbf{A}_n \nabla u) + \mathbf{B}_n \circ (I, u, \nabla u) = 0 \quad & \text{in } \Omega \\ u = \mathbf{g}_n \quad & \text{on } \Gamma \end{aligned} \tag{3.79}$$

with

$$\begin{gathered} \mathbf{B}_n(x, u, p) = B_{\theta_n} \circ (\theta_n(x), u, {}^*J_{\theta_n}^{-1}(x) p) - f_{\theta_n} \circ \theta_n(x) \\ \mathbf{A}_n = A^{\theta_n} \ \text{ and } \ \mathbf{g}_n = g_{\theta_n} \circ \theta_n \end{gathered} \tag{3.80}$$

The inequalities from the lemmas 29 and 30 yield the boundedness of the u_n in $\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R})$ for a $\beta > 0$ (see [32]).

The assumption **(A3)** yields for any open bounded subset U of $\overline{\Omega} \times \mathbb{R}^{n+1}$ the existence of a bound K_θ such that

$$\|B_{\theta'} \circ (\theta'(\cdot), \cdot, \cdot)\|_{\mathcal{C}^{0,1}(\overline{U};\mathbb{R})} \leq K_\theta$$

consequently for any n big enough

$$\|\mathbf{B}_n \circ (I, u_n, \nabla u_n) + f_{\theta_n} \circ \theta_n\|_{\mathcal{C}^{0,\gamma}} \leq \lambda K_\theta \|u_n\|_{\mathcal{C}^{1,\gamma}}$$

with $\gamma = \min(\beta, \alpha)$ for a $\lambda > 0$. Therefore, $\{\mathbf{B}_n \circ (I, u_n, \nabla u_n) + f_{\theta_n} \circ \theta_n,\ n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}^{0,\gamma/2}(\overline{\Omega};\mathbb{R})$. The sequence $(f_{\theta_n} \circ \theta_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R})$ and hence relatively compact $\mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R})$ as well as in $\mathcal{C}^{0,\gamma/2}(\overline{\Omega};\mathbb{R})$. Finally, $\{\mathbf{B}_n \circ (I, u_n, \nabla u_n),\ n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}^{0,\gamma/2}(\overline{\Omega};\mathbb{R})$. The sequence $(\mathbf{g}_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{C}^{2,\alpha}(\Gamma;\mathbb{R})$ and therefore relatively compact in $\mathcal{C}^{2,\gamma/2}(\Gamma;\mathbb{R})$. Finally, we have found some convergent sequences $(\mathbf{f}_n, \mathbf{g}_n) \in \mathcal{C}^{0,\gamma/2}(\overline{\Omega};\mathbb{R}) \times \mathcal{C}^{2,\gamma/2}(\Gamma;\mathbb{R})$ (with limit $(\mathbf{f}, \mathbf{g})$) such that a subsequence y_n of the u_n satisfies

$$\begin{aligned} \operatorname{div}(\mathbf{A}_n \nabla y_n) &= \mathbf{f}_n \quad \text{in } \Omega \\ y_n &= \mathbf{g}_n \quad \text{on } \Gamma \end{aligned} \tag{3.81}$$

At that point some results concerning the dependence of the operator $(\mathbf{f}_n, \mathbf{g}_n) \to y_n$ in $\mathbf{A}_n$ are needed. They are provided in the lemma 28 of the next section.

$$y_n = R_{A_n}(\mathbf{f}_n, \mathbf{g}_n) = R_A(\mathbf{f}_n, \mathbf{g}_n) - (R_A - R_{A_n})(\mathbf{f}_n, \mathbf{g}_n) \to R_A(\mathbf{f}, \mathbf{g}) \text{ in } \mathcal{C}^{2,\gamma/2}(\overline{\Omega};\mathbb{R})$$

and y_n converges in $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R})$. That implies that $\mathbf{B}_n \circ (I, y_n, \nabla y_n)$ converges to $B_\theta(\theta, y_n, {}^*J_\theta^{-1}\nabla y_n)$ in $\mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R})$ and we apply the same method again with the sequences that converge in $\mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R})$ and $\mathcal{C}^{2,\alpha}(\Gamma;\mathbb{R})$. We conclude that $y_n \to R_A(\mathbf{f}, \mathbf{g})$ in $\mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$. That concludes the proof. ■

3.2.3.1 Dependence of $[\operatorname{div}(A\nabla\cdot)]^{-1}$ w.r. to A

We associate to a $A \in \mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})$ such that (A2) holds the operator

$$R_A = [\operatorname{div}(A\nabla\cdot)]^{-1} \tag{3.82}$$

(see proposition 22 for the definition of this operator). For any such A, R_A is a linear continuous operator from $E = \mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Gamma;\mathbb{R})$ to $F = \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$.

Lemma 28 *Let A_n be a sequence of $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{n\times n})$ such that* (A2) *holds for any n.*

$$A_n \to A \text{ in } \mathcal{C}^{1,\alpha} \text{ implies } R_{A_n} \to R_A \text{ in } \mathcal{L}(E, F)$$

Proof – For any $(f, g) \in E$, $u = R_A(f, g)$ and $u_n = R_{A_n}(f, g)$ are solutions of

$$\left| \begin{aligned} \operatorname{div}(A\nabla u) &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned} \right. \quad \text{and} \quad \left| \begin{aligned} \operatorname{div}(A_n\nabla u_n) &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned} \right. \tag{3.83}$$

Let $K > 0$ be such that the inequality

$$\|R_B(f, g)\|_{\mathcal{C}^{2,\alpha}} \leq K[\|f\|_{\mathcal{C}^{0,\alpha}} + \|g\|_{\mathcal{C}^{2,\alpha}}] \tag{3.84}$$

holds for $B = A$ as well as the $B = A_n$ for $n \geq n_0$ (see the proposition 22). The mappings u and u_n satisfy $\mathrm{div}(A\nabla u - A_n \nabla u_n) = 0$ and it yields

$$\left| \begin{array}{rl} \mathrm{div}(A_n \nabla(u - u_n)) = -\mathrm{div}((A - A_n)\nabla u) & \text{in } \Omega \\ u - u_n = 0 & \text{on } \Gamma \end{array} \right. \tag{3.85}$$

The equation (3.84) implies the existence of a $\lambda > 0$ such that

$$\|u - u_n\|_{\mathcal{C}^{2,\alpha}} \leq \lambda K \|A - A_n\|_{\mathcal{C}^{1,\alpha}} \|u\|_{\mathcal{C}^{2,\alpha}} \leq \lambda K^2 \|A - A_n\|_{\mathcal{C}^{1,\alpha}} [\|f\|_{\mathcal{C}^{0,\alpha}} + \|g\|_{\mathcal{C}^{2,\alpha}}]$$

and therefore

$$\|R_A - R_{A_n}\|_{\mathcal{L}(E,F)} \leq \lambda K^2 \|A - A_n\|_{\mathcal{C}^{1,\alpha}} \to 0 \text{ when } A_n \to A$$

■

3.2.3.2 Uniform estimates for $\mathbf{A}_n$ and $\mathbf{B}_n$

Lemma 29 *There is a $n_0 \in \mathbb{N}$ and $\nu_- > 0$, $\nu_+ > 0$ such that $n \geq n_0$ implies that for any $\xi \in \mathbb{R}^n$*

$$\nu_- \cdot |\xi|^2 \leq {}^*\xi \cdot \mathbf{A}_n \cdot \xi \leq \nu_+ \cdot |\xi|^2 \tag{3.86}$$

Proof – We recall that $A^\theta = \gamma_\theta \, {}^* J_\theta^{-1} (A_\theta \circ \theta) J_\theta^{-1}$. Therefore, for any $\xi \in \mathbb{R}^n$, we have

$$\gamma_\theta \cdot \nu_-(\theta) |J_\theta^{-1} \xi|^2 \leq {}^*\xi \cdot A^\theta \cdot \xi \leq \gamma_\theta \cdot \nu_+(\theta) |J_\theta^{-1} \xi|^2$$

As $A^{\theta_n} \to A^\theta$ in $\mathcal{C}^0(\overline{\Omega}; \mathbb{R}^{n \times n})$, for any $\varepsilon > 0$, there is a $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies (3.86) with

$$\begin{aligned} \nu_- &= [\min_{x \in \overline{\Omega}} \gamma_\theta(x)][\min_{x \in \overline{\Omega}} \|J_\theta^{-1}\|^2] \cdot \nu_-(\theta) - \varepsilon \\ \nu_+ &= [\max_{x \in \overline{\Omega}} \gamma_\theta(x)][\max_{x \in \overline{\Omega}} \|J_\theta^{-1}\|^2] \cdot \nu_-(\theta) + \varepsilon \end{aligned} \tag{3.87}$$

A sufficiently small ε yields the result. ■

Lemma 30 *There is a $n_0 \in \mathbb{N}$, $\alpha > 0$, $\beta \geq 0$, and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $n \geq n_0$, $x \in \overline{\Omega}$, $u \in \mathbb{R}$ and $M \in \mathbb{R}_+$*

$$\mathbf{B}_n(x, u, 0) \cdot u \leq -\alpha u^2 + \beta \tag{3.88}$$

$$|u| \leq M \text{ implies } |\mathbf{B}_n(x, u, p) + \langle \mathrm{div}\, \mathbf{A}_n(x), p \rangle| \leq \gamma(M)(1 + |p|^2) \tag{3.89}$$

Proof – Let $M > 0$. As the mapping $\theta \mapsto f_\theta \circ \theta$ is continous, for any $n_0 \in \mathbb{N}$, there is a bound $\lambda_\theta > 0$ such that

$$n \geq n_0 \text{ implies } \|f_{\theta_n} \circ \theta_n\|_{\mathcal{C}^0} \leq \lambda_\theta \tag{3.90}$$

Due to the definition of $\mathbf{B}_n$ and to the local uniformity of $\alpha(\theta)$ and $\beta(\theta)$, there is a $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies

$$\mathbf{B}_n(x, u, 0) \cdot u \leq -\alpha(\theta) u^2 + \beta(\theta) + \lambda_\theta \cdot u$$

As

$$\lambda_\theta \cdot u \leq \frac{\lambda_\theta^2}{2\alpha(\theta)} + \frac{\alpha(\theta)}{2} u^2$$

we obtain

$$\mathbf{B}_n(x,u,0) \leq -\frac{\alpha(\theta)}{2}u^2 + \beta(\theta) + \frac{\lambda_\theta^2}{2\alpha(\theta)} \tag{3.91}$$

and (3.88) is proved. The local uniformity of $\gamma(\theta)$ implies that there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies

$$\forall\, x \in \overline{\Omega},\ |u| \leq M \ \text{ implies } \ |B_{\theta_n}(x,u,p) + \langle \operatorname{div}(A_{\theta_n}(x),p)\rangle\,| \leq \gamma(\theta)(M)(1+|p|^2)$$

therefore, under the same assumptions

$$|B_{\theta_n}(\theta_n(x),u,p) + \langle \operatorname{div}(A_{\theta_n} \circ \theta_n(x)),p)\rangle\,| \leq \gamma(\theta)(M)(1+|p|^2)$$

As $A_{\theta_n} \circ \theta_n = \gamma_{\theta_n}^{-1}\, {}^*J_{\theta_n} A^{\theta_n} J_{\theta_n} \to A^\theta$ in $\mathcal{C}^1(\overline{\Omega};\mathbb{R}^{n\times n})$. we have

$$|\mathbf{B}_n(x,u,,p) + \langle \operatorname{div}\mathbf{A}_n(x),p\rangle\,| \leq R_1(n,x,p) + R_2(n,x,p) + R_3(n,x,p)$$

where

$$\begin{array}{rcl} R_1(n,x,p) & := & |B_{\theta_n}(\theta_n(x),u,\,{}^*J_{\theta_n}^{-1}(x)p) + \left\langle \operatorname{div}(A_{\theta_n} \circ \theta_n(x)),\,{}^*J_{\theta_n}^{-1}(x)p\right\rangle| \\ & \leq & \kappa_\theta\gamma(\theta)(M)(1+|p|^2) \ \text{ with } \ \kappa_\theta := \max(1,\|J_\theta^{-1}\|^2) \end{array}$$

$$\begin{array}{rcl} R_2(n,x,p) & := & |\left\langle \operatorname{div}(A^{\theta_n}(x)),p\right\rangle - \left\langle J_{\theta_n}^{-1}(x)\cdot \operatorname{div}(A_{\theta_n} \circ \theta_n(x)),p\right\rangle| \\ & \leq & \mu_\theta|p| \ \text{ with } \ \mu_\theta := \|\operatorname{div}(A^{\theta_n}) - J_{\theta_n}^{-1}\cdot \operatorname{div}(A_{\theta_n} \circ \theta_n)\|_{\mathcal{C}^0} \end{array}$$

and $R_3(n,x,p) := |f_{\theta_n} \circ \theta_n(x)| \leq \lambda_\theta$ (see (3.90)). We finally end up with

$$|\mathbf{B}_n(x,u,,p) + \langle \operatorname{div}\mathbf{A}_n(x),p\rangle\,| \leq \gamma(M)(1+|p|^2) \tag{3.92}$$

with

$$\gamma(x) = \kappa_\theta\gamma(\theta)(x) + \frac{\mu_\theta}{2} + \lambda_\theta \tag{3.93}$$

■

3.2.4 Assumption (ii) : Regularity

Proposition 35 *The theorem 5 holds when the condition* (ii) *is replaced with*

(ii') *For any* $(x,y) \in U \times V$ *such that* $\mathcal{A}(x,y) = y$, $\xi = 0 \in F'$ *is the unique solution of the system*

$$\left\{ \begin{array}{c} \xi \in \operatorname{Ker}[\partial_x\mathcal{A}(x,\mu)]^* \\ \forall\, \nu \in M,\ \langle \xi, \partial_\mu\mathcal{A}(x,\mu)\cdot\nu\rangle_{F'\times F} = 0 \end{array} \right. \tag{3.94}$$

Proof – We have to show that (ii') implies that for any $f \in F$, there is a $(\zeta,\nu) \in E \times M$ such that $\partial_x\mathcal{A}(x,\mu)\cdot\zeta + \partial_\mu\mathcal{A}(x,\mu)\cdot\nu = f$ or equivalently

$$\partial_x\mathcal{A}(x,\mu)\cdot\zeta = f - \partial_\mu\mathcal{A}(x,\mu)\cdot\nu$$

As $\partial_x\mathcal{A}(x,\mu)$ is a Fredholm operator, this is equivalent to

$$\forall\, f \in F,\ \exists\, \nu \in M,\ \xi \in \operatorname{Ker}[\partial_x\mathcal{A}(x,\mu)]^* \Rightarrow \langle \xi, f - \partial_\mu\mathcal{A}(x,\mu)\cdot\nu\rangle = 0 \tag{3.95}$$

Let $(\xi_i)_{i\in\{1,...n\}}$ be a basis of $\mathrm{Ker}\,[\partial_x \mathcal{A}(x,\mu)]^*$. Then, (3.95) is equivalent to

$$\forall\, f \in F,\ \exists\, \nu \in M,\ \forall\, i \in \{1,...,n\},\ \langle \xi_i, f - \partial_\mu \mathcal{A}(x,\mu)\cdot \nu\rangle = 0$$

and this proposition holds when

$$\forall\,(f_1,...,f_n) \in \mathbb{R}^n,\ \exists\, \nu \in M,\ \forall\, i \in \{1,...,n\},\ \langle L_i, \nu\rangle = f_i \tag{3.96}$$

$$\text{with } L_i \in M' \text{ defined by } \langle L_i,\nu\rangle_{M'\times M} = \langle \xi_i, \partial_\mu \mathcal{A}(x,\mu)\cdot\nu\rangle_{F'\times F} \tag{3.97}$$

If the proposition (3.96) is satisfied then the family $(L_i)_{i\in\{1,...,n\}}$ is free because for any $\lambda = (\lambda_1,...,\lambda_n) \in \mathbb{R}^n$ such that $\lambda \neq 0$, there is a ν such that $\langle \sum_{i=1}^n \lambda_i L_i, \nu\rangle = 1$. Conversely, as $H = \{(f_1,...,f_n) \in \mathbb{R}^n,\ \exists\,\nu \in M,\ \forall\, i \in \{1,...,n\}\ < L_i, \nu >= f_i\}$ is a vectorial subspace of $\mathbb{R}^n$, if (3.96) is not satisfied, there is a $\lambda \neq 0 \in H^\perp$: for any $\nu \in M$, $\sum_{i=1}^n \lambda_i \langle L_i,\nu\rangle = 0$ and therefore $\sum_{i=1}^n \lambda_i L_i = 0$. Finally, (3.96) is equivalent to the fact that the family $(L_i)_{i\in\{1,...,n\}}$ is free i.e.

$$\forall\,\nu \in M,\ \left\langle \sum \lambda_i \xi_i, \partial_\mu \mathcal{A}(x,\mu)\cdot\nu\right\rangle = 0 \Longrightarrow \lambda = 0$$

that is

$$\xi \in \mathrm{Ker}\,[\partial_x \mathcal{A}(x,\mu)]^* \text{ and } \forall\,\nu \in M,\ \langle \xi, \partial_\mu \mathcal{A}(x,\mu)\cdot\nu\rangle = 0 \Longrightarrow \xi = 0$$

■

An easy consequence of that proposition is the

Corollary 5 *Let $\bar{E}$ and $\bar{F}$ be two Hilbert spaces such that $E \hookrightarrow \bar{E}$ and $F \hookrightarrow \bar{F}$. We assume that there is a unique extension $\overline{\partial_x \mathcal{A}(x,\mu)}$ of $\partial_x \mathcal{A}(x,\mu)$ to $\mathcal{L}(\bar{E},\bar{F})$.*

The theorem 5 still holds when the condition (ii) *is replaced with*

(ii″) *For any $(x,y) \in U \times V$ such that $\mathcal{A}(x,y) = y$, $\xi = 0 \in \bar{F}'$ is the unique solution of the system*

$$\left\{\begin{array}{c} \xi \in \mathrm{Ker}\,[\overline{\partial_x \mathcal{A}(x,\mu)}]^* \\ \forall\,\nu \in M,\ \langle \xi, \partial_\mu \mathcal{A}(x,\mu)\cdot\nu\rangle_{\bar{F}'\times\bar{F}} = 0 \end{array}\right. \tag{3.98}$$

Proposition 36 *The value 0 is regular for E^Ω if and only if for any $\Omega' \in \mathcal{D}$, and any $u \in$ such that $E^{\Omega'}(I,u) = 0$, the mapping $dE^{\Omega'}(I,u)$ is onto.*

Proof – The value 0 is regular for E^Ω if and only if $E^\Omega(\theta,u) = 0$ implies that for any $(f,g) \in \mathcal{C}^{0,\alpha}(\overline{\Omega};\mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Gamma;\mathbb{R})$ there is a $(\delta\theta, y) \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n) \times \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$ such that

$$\partial_\theta E^\Omega(\theta,u)\cdot\delta\theta + \partial_u E^\Omega(\theta,u)\cdot y = (f,g) \tag{3.99}$$

The equation $E^\Omega(\theta,u) = 0$ is equivalent to $E^{\theta(\Omega)}(I,\phi_\theta(u)) = 0$ (see the equation (3.104)). On the other hand, (3.99) is equivalent to the system

$$\begin{array}{c} \partial_\theta E_1^\Omega(\theta,u)\cdot\delta\theta + \partial_u E_1^\Omega(\theta,u)\cdot y = f \\ \partial_\theta E_2^\Omega(\theta,u)\cdot\delta\theta + \partial_u E_2^\Omega(\theta,u)\cdot y = g \end{array} \tag{3.100}$$

wich is itself equivalent to

$$\begin{array}{l} \psi_\theta \circ \left[\partial_\theta E_1^{\theta(\Omega)}(I,\phi_\theta(u))\cdot\phi_\theta(\delta\theta) + \partial_u E_1^{\theta(\Omega)}(I,\phi_\theta(u))\cdot\phi_\theta(y)\right] = f \\ \psi_\theta \circ \left[\partial_\theta E_2^{\theta(\Omega)}(I,\phi_\theta(u))\cdot\phi_\theta(\delta\theta) + \partial_u E_2^{\theta(\Omega)}(I,\phi_\theta(u))\cdot\phi_\theta(y)\right] = g \end{array} \tag{3.101}$$

(see (3.105) et (3.106) for the first equation, (3.44) for the second) or

$$\psi_\theta \circ \left[\partial_\theta E^{\theta(\Omega)}(I, \phi_\theta(u)) \cdot \phi_\theta(\delta\theta) + \partial_u E^{\theta(\Omega)}(I, \phi_\theta(u)) \cdot \phi_\theta(y)\right] = (f, g) \quad (3.102)$$

as ψ_θ is an isomorphism from $\mathcal{C}^{0,\alpha}(\overline{\Omega_\theta}; \mathbb{R})$ to $\mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R})$ and from $\mathcal{C}^{2,\alpha}(\Gamma_\theta; \mathbb{R})$ to $\mathcal{C}^{2,\alpha}(\Gamma; \mathbb{R})$ and ϕ_θ from $\mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$ to $\mathcal{C}^{2,\alpha}(\overline{\Omega_\theta}; \mathbb{R})$ and from $\mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ to $\mathcal{C}^{2,\alpha}(\overline{\Omega_\theta}; \mathbb{R}^n)$, the regularity requirement is equivalent to : for any (θ, u') such that $E^{\theta(\Omega)}(I, u') = 0$, there is a solution $((\delta\theta)', y')$ to

$$\partial_\theta E^{\theta(\Omega)}(I, u') \cdot (\delta\theta)' + \partial_u E^{\theta(\Omega)}(I, u') \cdot y' = (f', g') \quad (3.103)$$

■

Lemma 31 *For any $\Omega \in \mathcal{D}$, $\theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$, $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$, $\delta\theta \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ and $y \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$ we have*

$$E_1^\Omega(\theta, u) = \psi_\theta \circ E_1^{\theta(\Omega)} \circ \phi_\theta(\theta, u) \quad (3.104)$$

$$\partial_\theta E_1^\Omega(\theta, u) \cdot \delta\theta = \psi_\theta \circ \partial_\theta E_1^{\theta(\Omega)}(I, \phi_\theta(u)) \cdot \phi_\theta(\delta\theta) \quad (3.105)$$

$$\partial_u E_1^\Omega(\theta, u) \cdot y = \psi_\theta \circ \partial_u E_1^{\theta(\Omega)}(I, \phi_\theta(u)) \circ \phi_\theta(y) \quad (3.106)$$

Proof – The relation (3.106) is a direct consequence of the definition of E_1. The proof of (3.105) is more involved.

As in the proposition 28, we define for any $\theta_0 \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$ the linear continuous operator $S_n : \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n) \to \mathcal{C}^{2,\alpha}(\overline{\Omega_{\theta_0}}; \mathbb{R}^n)$ by

$$S_n(f) = f \circ \theta_0^{-1} \text{ or } S_n^{-1}(f) = f \circ \theta_0 \quad (3.107)$$

We set $L_{\theta(\Omega)} = \mathrm{div}(A_\theta \nabla \cdot) + B_\theta \circ (I, \cdot, \nabla \cdot) - f_\theta$. We have $\psi_\theta \circ L_{\theta(\Omega)} \circ \phi_\theta(u) = [L_{\theta(\Omega)}(u \circ \theta^{-1})] \circ \theta$ and therefore

$$\begin{aligned}
\psi_\theta \circ L_{\theta(\Omega)} \circ \phi_\theta(u) &= [L_{(\theta\circ\theta_0^{-1})\circ\theta_0(\Omega)}(u \circ \theta_0^{-1} \circ (\theta_0 \circ \theta^{-1}))] \circ (\theta \circ \theta_0^{-1}) \circ \theta_0 \\
&= [L_{S_n(\theta)\circ[\theta_0(\Omega)]}(u \circ \theta_0^{-1} \circ [S_n(\theta)]^{-1})] \circ S_n(\theta) \circ \theta_0 \\
&= [[\psi_\delta \circ L_{\delta(\theta_0(\Omega))} \circ \phi_\delta]|_{\delta=S_n(\theta)}(u \circ \theta_0^{-1})] \circ \theta_0 \\
&= [\psi_{\theta_0} \circ [\psi_\delta \circ L_{\delta(\theta_0(\Omega))} \circ \phi_\delta]|_{\delta=S_n(\theta)} \circ \phi_{\theta_0}](u)
\end{aligned}$$

or equivalently $E_1^\Omega(\theta, u) = \psi_{\theta_0} \circ E_1^{\theta_0(\Omega)} \circ \phi_{\theta_0}(\theta, u)$. Consequently we obtain $\partial_\theta E_1^\Omega(\theta, u) \cdot \delta\theta = \psi_\theta \circ \partial_\theta E_1^{\theta(\Omega)}(I, \phi_\theta(u)) \cdot (\delta\theta \circ \theta^{-1})$ ■

3.2.5 Computations of the derivatives of E

Let $N = (N_1, N_2)$ be the unique extension of $\partial_u E(I, u)$ as an operator of $H^1(\Omega; \mathbb{R})$ to $H^{-1}(\Omega; \mathbb{R}) \times H^{1/2}(\Gamma; \mathbb{R})$.

Lemma 32 *Let $\xi = (\xi_1, \xi_2) \in H_0^1(\Omega; \mathbb{R}) \times H^{-1/2}(\Gamma; \mathbb{R})$. We set*

$$K = -\nabla_p B(I, u, \nabla u) \quad (3.108)$$

$$L = \partial_u B(I, u, \nabla u) - \mathrm{div}\, K \quad (3.109)$$

We have $\xi \in \mathrm{Ker}\, N^*$ iff $\xi_1 \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R})$, $\xi_2 \in \mathcal{C}^{1,\alpha}(\Gamma; \mathbb{R})$ and ξ is solution of

$$\begin{cases} \mathrm{div}(A\nabla\xi_1) + L \cdot \xi_1 + \langle K, \nabla\xi_1 \rangle = 0 & \text{in } \Omega \\ \xi_2 = \langle A\nabla\xi_1, n \rangle & \text{on } \Gamma \end{cases} \quad (3.110)$$

Proof – For any $\phi \in H^1(\Omega;\mathbb{R})$, we have

$$N \cdot \phi = \left(\mathrm{div}(A\nabla\phi) + \partial_u B \cdot \phi + \langle \nabla_p B, \nabla\phi\rangle, \phi|_\Gamma\right)$$

Let $\xi \in \mathrm{Ker}\, N^*$. For any $\phi \in H^1(\Omega;\mathbb{R})$, we have

$$\langle N^* \cdot \xi, \phi\rangle = \langle \xi, N \cdot \phi\rangle = \langle \xi_1, N_1 \cdot \phi\rangle + \langle \xi_2, N_2 \cdot \phi\rangle = 0 \tag{3.111}$$

In particular it is true for any $\phi \in H_0^1(\Omega;\mathbb{R})$. As for any such ϕ

$$\langle \xi_1, \mathrm{div}(A\nabla\phi)\rangle = \langle \mathrm{div}(A\nabla\xi_1), \phi\rangle \tag{3.112}$$

$$\langle \xi_1, \partial_u B \cdot \phi\rangle = \langle \partial_u B \cdot \xi_1, \phi\rangle \tag{3.113}$$

and

$$\langle \xi_1, \langle \nabla_p B, \nabla\phi\rangle\rangle = -\langle \mathrm{div}(\xi_1 \nabla_p B), \phi\rangle = -\langle \xi_1 \mathrm{div}(\nabla_p B) + \langle \nabla\xi_1, \nabla_p B\rangle, \phi\rangle \tag{3.114}$$

we deduce that ξ_1 solution of

$$\begin{cases} \mathrm{div}(A\nabla\xi_1) + L \cdot \xi_1 + \langle K, \nabla\xi_1\rangle = 0 & \text{in } \Omega \\ \xi_1 = 0 & \text{on } \Gamma \end{cases}$$

and therefore $\xi_1 \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$. We go back to (3.111) but with a general $\phi \in H^1(\Omega;\mathbb{R})$. For such a ϕ, (3.112) becomes

$$\langle \xi_1, \mathrm{div}(A\nabla\phi)\rangle = \langle \mathrm{div}(A\nabla\xi_1), \phi\rangle - \langle \langle A\nabla\xi_1, n\rangle, \phi\rangle$$

and (3.113) and (3.114) do not change. Finally, for any ϕ,

$$\langle \xi, N \cdot \phi\rangle = -\langle \langle A\nabla\xi_1, n\rangle, \phi\rangle + \langle \xi_2, \phi\rangle = 0$$

and the condition $\xi_2 = \langle A\nabla\xi_1, n\rangle$ is proved and consequently the regularity of ξ_2. The proof of the converse statement is simple. ■

For any $h > 0$, we define the set $V_h(\Gamma)$ by

$$V_h(\Gamma) = \{x \in \overline{\Omega},\ d(x,\Gamma) < h\} \tag{3.115}$$

For a small enough h, the projection $p : V_h(\Gamma) \to \Gamma$ on Γ is uniquely defined. Let ξ_m, $m \in \mathbb{N}^*$, be a sequence of smooth mappings $\overline{\Omega} \to \mathbb{R}$ such that

$$\xi_m(x) \begin{cases} = 0 & \text{if } x \in \Gamma \\ \in [0,1] & \text{if } x \in V_{1/m}(\Gamma) \\ = 1 & \text{if } x \in \Omega - V_{1/m}(\Gamma) \end{cases} \tag{3.116}$$

and moreover

$$\nabla\xi_m - \frac{\chi_{V_{1/m}(\Gamma)}}{m} \cdot n \circ p \to 0 \ \text{ in } \ L^1(\Omega;\mathbb{R}^n) \tag{3.117}$$

Definition 16 *For any $\phi \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$ and $m \in \mathbb{N}^*$, we set*

$$[\phi]_m := \xi_m \cdot \phi \tag{3.118}$$

We define the set $\mathcal{D}$ of $(\mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}))'$ by

$$\zeta \in \mathcal{D} \ \text{ iff } \ \forall\, \phi \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}),\ \langle \zeta, [\phi]_m\rangle \to \langle \zeta, \phi\rangle \tag{3.119}$$

and $\mathcal{D}_n$ as the subset of $(\mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n))'$ defined by $\mathcal{D}_1 = \mathcal{D}$ and $\mathcal{D}_n = \mathcal{D}_{n-1} \times \mathcal{D}$. For any $V = (V_1, ..., V_n) \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$, we set

$$[V]_m = ([V_1]_m, ..., [V_n]_m) \tag{3.120}$$

As for any $\lambda \in \mathbb{R}$, $\lambda[\phi]_m = [\lambda\phi]_m$, the set $\mathcal{D}$ is a linear subspace of $(\mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}))'$.

Lemma 33 *Let $f \in \mathcal{C}^0(\overline{\Omega};\mathbb{R})$ and $i \in \{1,...,n\}$. The mapping*

$$\zeta : \phi \mapsto \int_\Omega f\partial_i\phi\,dx - \int_\Gamma f\phi n_i\,d\mathcal{H}^{n-1} \tag{3.121}$$

belongs to $\mathcal{D}$.

Proof – For any $\phi \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R})$ and $m \in \mathbb{N}^*$, as $[\phi]_m = 0$ on Γ, we have

$$\langle\zeta,[\phi]_m\rangle = \int_\Omega f\partial_i[\phi]_m\,dx$$

and therefore, for any $h>0$,

$$\langle\zeta,[\phi]_m\rangle = \int_\Omega f\partial_i\phi\,dx + \int_{\Omega-V_h(\Gamma)} f\partial_i([\phi]_m-\phi)\,dx - \int_{V_h(\Gamma)} f\partial_i(\phi-[\phi]_m)\,dx$$

For any $h>0$

$$\lim_{m\to+\infty}\int_{\Omega-V_h(\Gamma)} f\partial_i([\phi]_m-\phi)\,dx = 0$$

Moreover, as $\partial_i[\phi]_m = \xi_m\partial_i\phi + \partial_i\xi_m\phi$, we have

$$\begin{aligned}\int_{V_h(\Gamma)} f\partial_i(\phi-[\phi]_m)\,dx &= \int_{V_h(\Gamma)} f(1-\xi_m)\partial_i\phi\,dx + \int_{V_h(\Gamma)} f\phi\partial_i\xi_m\,dx\\ &= \varepsilon_m + \frac{1}{m}\int_{V_{1/m}(\Gamma)} f\phi n_i\circ p\,dx\end{aligned}$$

with $\varepsilon_m \to 0$ when $m\to+\infty$. Therefore,

$$\int_{V_h(\Gamma)} f\partial_i(\phi-[\phi]_m)\,dx \to \int_\Gamma f\phi n_i\,d\mathcal{H}^{n-1} \text{ when } m\to+\infty$$

(see Federer's formula [JPZ+DELF: Shells th 2.4, p. 66]). Finally,

$$\langle\zeta,[\phi]_m\rangle \to \int_\Omega f\partial_i\phi\,dx - \int_\Gamma f\phi n_i\,d\mathcal{H}^{n-1} = \langle\zeta,\phi\rangle$$

■

The following corollary is obvious.

Corollary 6 *Let $F \in \mathcal{C}^0(\overline{\Omega};\mathbb{R}^{n\times n})$. The mapping*

$$\phi \mapsto \int_\Omega F\cdot\cdot D\phi\,dx - \int_\Gamma \langle F\cdot n,\phi\rangle\;d\mathcal{H}^{n-1} \tag{3.122}$$

belongs to $\mathcal{D}_n$.

Proposition 37 *Let $\Omega \in \mathcal{D}$. For any $\xi \in \mathrm{Ker}\,N^*$ and any $V \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$, we have*

$$\langle\xi,\partial_\theta E(I,u)\cdot V\rangle = \langle d_\xi,V\rangle + \int_\Gamma \langle b_\xi,V\rangle_{\mathbb{R}^n}\;d\mathcal{H}^{n-1} \tag{3.123}$$

with $d_\xi \in \mathcal{D}_n$ and

$$b_\xi \;=\; \left[-(n\otimes A\nabla u) + (A\nabla u\otimes n) + \frac{\partial u}{\partial n}A - \dot g\otimes An\right]\nabla\xi_1 \tag{3.124}$$

Proof – Il faur régler l'histoire du

(i) First, we compute $\langle \xi_1, \partial_\theta E_1(I,u)\cdot V\rangle$. It is classical that

$$\partial_\theta J_\theta|_{\theta=I}\cdot V = DV \text{ and } \partial_\theta \gamma_\theta|_{\theta=I}\cdot V = \operatorname{div} V$$

Consequently,

$$K(V) := \partial_\theta A^\theta|_{\theta=I}\cdot V = \operatorname{div} V\cdot A - {}^t DVA - ADV + \dot{A}\cdot V$$

Therefore,

$$\partial_\theta[\gamma_\theta^{-1}\operatorname{div}(A^\theta u)]_{\theta=I}\cdot V = -(\operatorname{div} V)\operatorname{div}(A\nabla u) + \operatorname{div}(K(V)\nabla u)$$

As

$$\langle -(\operatorname{div} V)\operatorname{div}(A\nabla u), \xi_1\rangle = \int_\Omega f_\xi\cdot \operatorname{div} V\, dx \text{ with } f_\xi\in\mathcal{C}^0(\overline{\Omega};\mathbb{R}) \text{ and } f_\xi = 0 \text{ on } \Gamma$$

we have

$$V\mapsto \langle -(\operatorname{div} V)\operatorname{div}(A\nabla u), \xi_1\rangle \in \mathcal{D}_n \tag{3.125}$$

(see the corollary 6). On the other hand

$$\langle \operatorname{div}(K(V)\nabla u), \xi_1\rangle = -\langle K(V)\nabla u, \nabla\xi_1\rangle$$

and then

$$\begin{aligned}-\langle K(V)\nabla u, \nabla\xi_1\rangle &= -\int_\Gamma \langle A\nabla u, \nabla\xi_1\rangle\langle V, n\rangle\, d\mathcal{H}^{n-1} + \int_\Gamma \frac{\partial\xi_1}{\partial n}\langle A\nabla u, V\rangle\, d\mathcal{H}^{n-1}\\ &\quad + \int_\Gamma \frac{\partial u}{\partial n}\langle A\nabla\xi_1, V\rangle\, d\mathcal{H}^{n-1} + \langle g_\xi, V\rangle\\ &\quad -\langle[\dot{A}\cdot V]\cdot\nabla u, \nabla\xi_1\rangle\end{aligned}$$

where $g_\xi\in\mathcal{D}_n$. The mapping $V\mapsto\langle[\dot{A}\cdot V]\cdot\nabla u, \nabla\xi_1\rangle$ also belongs to $\mathcal{D}_n$.

$$\partial_\theta B_\theta\circ(\theta, u, {}^*J_\theta^{-1}\nabla u)|_{\theta=I} = [\dot{B}(I,u,\nabla u)]\cdot V - [\partial_p B(I,u,\nabla u)]\cdot {}^*DV\cdot\nabla u$$

and therefore $V\mapsto\left\langle \partial_\theta B_\theta\circ(\theta,u,{}^*J_\theta^{-1}\nabla u)\cdot V, \xi\right\rangle\in\mathcal{D}_n$. We have also

$$\partial_\theta f_\theta\circ\phi_\theta|_{\theta=I} = [V\mapsto \dot{f}\cdot V]\in\mathcal{D}_n$$

(ii) We now compute $\langle\xi_2, \partial_\theta E_2(I,u)\cdot V\rangle$.

$$\langle\xi_2, \partial_\theta E_2(I,u)\cdot V\rangle = -\langle\xi_2, \dot{g}\cdot V\rangle = -\int_\Gamma \langle\langle A\nabla\xi_1, n\rangle\dot{g}, V\rangle\, d\mathcal{H}^{n-1}$$

■

Proposition 38 *Let $\Omega\in\mathcal{D}$ and $\xi\in\operatorname{Ker} N^*$. If for any $V\in\mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$, we have $\langle\xi, \partial_\theta E(I,u)\cdot V\rangle = 0$ then*

$$\left[\frac{\partial u}{\partial n} - \langle\dot{g}, n\rangle\right]\frac{\partial\xi_1}{\partial n} = 0 \text{ on } \Gamma \tag{3.126}$$

Proof – We use the equation (3.123). As for any $V \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$, $\langle \xi, \partial_\theta E(I,u)\cdot V\rangle = 0$, we have for any $m \in \mathbb{N}^*$ $\langle d_\xi, [V]_m\rangle = 0$ and therefore as $d_\xi \in \mathcal{D}_n$, $\langle d_\xi, V\rangle = 0$ Therefore, for any $V \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$

$$\int_\Gamma \langle b_\xi, V\rangle_{\mathbb{R}^n}\, d\mathcal{H}^{n-1} = 0$$

From the expression of b_ξ (see (3.124)), we deduce that for any $V \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^n)$

$$-\langle A\nabla u, \nabla\xi_1\rangle V_n + \frac{\partial \xi_1}{\partial n}\langle A\nabla u, V\rangle + \frac{\partial u}{\partial n}\langle A\nabla\xi_1, V\rangle - \langle An, \nabla\xi_1\rangle\langle \dot g, V\rangle = 0 \tag{3.127}$$

with $V_n = \langle V, n\rangle$. On the boundary, the following decompositions hold

$$\nabla\xi = \frac{\partial\xi}{\partial n}n\ ,\ \nabla u = \frac{\partial u}{\partial n}n + \nabla_\tau g \ \text{ and } \ V = V_\tau + V_n\cdot n$$

and therefore, (3.127) yields in particular when $V_\tau = 0$

$$\begin{aligned}
&-\left\langle A\nabla u, \frac{\partial\xi_1}{\partial n}n\right\rangle V_n + \frac{\partial\xi_1}{\partial n}\langle A\nabla u, V_n\cdot n\rangle \\
&+\frac{\partial u}{\partial n}\left\langle A\frac{\partial \xi_1}{\partial n}\cdot n, V_n\cdot n\right\rangle - \left\langle An, \frac{\partial\xi_1}{\partial n}\cdot n\right\rangle\langle \dot g, V_n\cdot n\rangle \quad = \quad 0
\end{aligned}$$

that is

$$\left[\frac{\partial u}{\partial n} - \langle \dot g, n\rangle\right]\frac{\partial \xi_1}{\partial n}\left[\langle An, n\rangle V_n\right] = 0 \tag{3.128}$$

As $A \geq 0$, this equation easily yields (3.126). ■

3.3 Application

Proposition 39 *Let the assumptions of the proposition 31 and **(A1)**, **(A2)**, **(A3)** be satisfied. Let Ω in $\mathcal{D}$ and*

$$\mathcal{O} = \{\Omega_\theta,\ \theta \in \mathrm{Diff}_{2,\alpha}(\overline{\Omega})\}$$

Assume that for any $\Omega \in \mathcal{O}$, there is a neighbourhood $V \subset \mathrm{Diff}_{2,\alpha}(\overline{\Omega})$ of the identity, an open set U of $\mathbb{R}^n$ and a function $G : \mathbb{R}^n \to \mathbb{R}$ such that for any $\theta \in V$, $g_\theta = G|_{\Gamma_\theta}$, and for any connected component C of Ω, $U \cap \partial C \neq \varnothing$ and

$$G \equiv 0 \ \text{ on } \ U \ \text{ but } \ G \not\equiv 0 \ \text{ on } \ \partial C \tag{3.129}$$

We assume moreover that the quasilinear equation is such that the shape-dependent functions B and f satisfy on any admissible set

$$B(\cdot,0,0) = 0 \ \text{ and } \ f = 0 \tag{3.130}$$

Then the state u of the quasilinear equation is shape differentiable in a dense open subset of the admissible geometries.

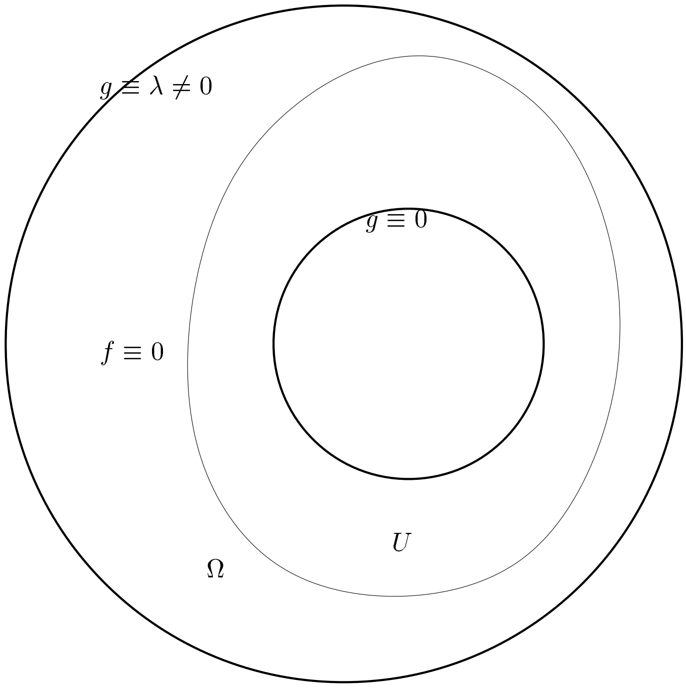


Figure 3.1: A typical data set for a shape-dependent quasilinear problem.

Proof – Let $\Omega \in \mathcal{O}$. As $G \equiv 0$ on U, for any connected component C of Ω, we have

$$\dot{g} = \nabla G = 0 \text{ on } U \cap \partial C$$

Therefore for any $\xi \in \operatorname{Ker} N^*$ such that for all $V \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^n)$, $\langle \xi, \partial_\theta E(I, u) \cdot V \rangle = 0$, by the proposition 38 there is a relatively open subset ω of $U \cap \partial C$ such that

$$(1) \ \frac{\partial u}{\partial n} \equiv 0 \text{ on } \omega \text{ or } (2) \ \frac{\partial \xi_1}{\partial n} \equiv 0 \text{ on } \omega$$

In the case (1), we have

$$\begin{aligned} \operatorname{div}(A(x)\nabla u) + B(x, u, \nabla u) &= 0 \quad \text{in } C \\ u &= 0 \quad \text{on } \omega \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on } \omega \end{aligned} \tag{3.131}$$

Standard continuation results (see [39]) yield therefore $u \equiv 0$ in C. But this is a contradiction with the non-identically zero boundary data. We would have $G \equiv 0$ on ∂C which is a contradiction with the assumptions. We are therefore in the case (2). As ξ_1 is a solution equal to zero on the boundary of (3.110) the same continuation result yield $\xi = 0$. The assumption (ii") of the corollary 5 is satisfied and the theorem 5 may be applied: there exists an open bounded set of $\mathcal{O}$ on which $\partial_u E$ is regular. The theorem 4 yields then the generic differentiability on that set. ■

Analyse de forme des équations de Navier-Stokes. Cas dynamique à frontière mobile

On s'intéresse désormais au système *dynamique* des équations de Navier-Stokes, dans un domaine spatio-temporel Q potentiellement non-cylindrique, c'est-à-dire non nécessairement de la forme $(0,T) \times \Omega$. La frontière à l'instant t du domaine spatial Ω_t tel que $\{t\} \times \Omega_t = Q \cap (\{t\} \times \mathbb{R}^3)$ est donc a priori mobile.

Ensembles d'évolution. On commence par caractériser la classe d'ensemble Q qui va convenir à nos fins : ce sont les ensembles d'évolution, obtenus par troncature d'ouverts spatio-temporels $\mathcal{C}^1$ entre les hyperplans $t = 0$ et $t = T$. Une hypothèse supplémentaire sur le champ des normales à cet ensemble permet d'affirmer que les domaines Ω_t pourront être reconstruits à partir du domaine Ω_0 comme image du flot associé à un champ de vitesse ad hoc.

Equations de Navier-Stokes à frontière mobile. Soit $\Sigma = \overline{\partial Q \cap ((0,T) \times \mathbb{R}^3)}$ la frontière latérale de Q. Avant toute chose, on détermine quel sens on peut donner au terme "solution" du système suivant:

$$\begin{gathered} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \text{ in } Q \text{ et } u(0) = u_0 \text{ dans } \Omega_0 \\ \operatorname{div} u = 0 \text{ in } Q \text{ et } u = 0 \text{ sur } \Sigma \end{gathered} \tag{3.132}$$

Le domaine Q étant non-cylindrique, il convient de choisir les classes de fonctions test adaptées et d'étudier leur rapport avec les espaces de fonctions test traditionnelles. On montre ensuite l'existence d'une telle solution par des méthodes de pénalisation.

Sensibilité par rapport à la géométrie. L'analyse porte à nouveau sur la dépendance des solutions du système par rapport à la forme du domaine Ω. Sous des hypothèses de régularité des données et de la solution nominale de l'équation de Navier-Stokes, on montre le caractère continuement différentiable de la vitesse par rapport aux perturbations de la géométrie dans les espaces adaptés.

Chapter 4

Shape Analysis of the NSE with Moving Boundaries

4.1 Geometry and Integration

4.1.1 Geometry and Structure of the Time-Space Sets

Positive colinearity. We introduce the following shortcut :

Notation 1 *For two vectors x and y of $\mathbb{R}^m$, we use the notation*

$$x \propto y$$

to express the fact that there exists a $\lambda > 0$ such that $x = \lambda y$.

Spatial and spatio-temporal normal. Assume that the open bounded set $M \subset \mathbb{R}^{n+1}$ is Lipschitz. The outward normal to M, denoted ν^M or simply ν, is defined for almost every $(t, x) \in \partial M$ and may be decomposed into its *temporal* and *spatial components* :

$$\nu = (\nu_t, \nu_x) \in \mathbb{R} \times \mathbb{R}^n \tag{4.1}$$

When $\nu_x(t, x) \neq 0$, we may define

$$n(t, x) = \frac{\nu_x(t, x)}{\|\nu_x(t, x)\|} \tag{4.2}$$

The unique value $v(t, x) \in \mathbb{R}$ such that

$$\nu = \frac{1}{\sqrt{1 + v^2}}(-v, n) \tag{4.3}$$

is the *normal spatial velocity*. Its value is given explicitely by the formula

$$v = -\frac{\nu_t}{\|\nu_x\|} \tag{4.4}$$

Definition 17 *A non-empty open subset Q of $\mathbb{R}^{n+1}$ is an* evolution set*, associated to the time interval $[0, T]$ if there is a $\mathcal{C}^1$ open bounded subset of $\mathbb{R}^{n+1}$ M such that*

$$Q = M \cap ((0, T) \times \mathbb{R}^n) \tag{4.5}$$

and

$$\forall (t,x) \in \partial M \cap \left([0,T] \times \mathbb{R}^N\right),\ \nu_x(t,x) \neq 0 \tag{4.6}$$

The domain M may be decomposed by a "sweeping" process. For any $t \in \mathbb{R}$, we define the sets Ω_t and Γ_t of $\mathbb{R}^n$ by

$$\{t\} \times \Omega_t = Q \cap (\{t\} \times \mathbb{R}^n) \text{ and } \Gamma_t = \partial\Omega_t$$

Clearly, for any $t \in (0,T)$, these sets depend only of M through Q. This is still true for $t \in \{0,T\}$ as a consequence of the following property:

Proposition 40 *Let $\tau = 0$ or $\tau = T$. Let $\mathcal{K}\text{-}\lim$ be the notation for the Kuratowski limit of sets*[1]*. We have*

$$\overline{\Omega_\tau} = \mathcal{K}\text{-}\lim_{t\to\tau} \Omega_t \tag{4.9}$$

Proof – Let $x \in \overline{\Omega_\tau}$. For any $\varepsilon > 0$, there is a $x' \in \Omega_\tau$ such that $\|x' - x\| \leq \varepsilon$. Moreover, as Q is open, for $|t-\tau|$ small enough, $d(x', \Omega_t) = 0$. Therefore, $\overline{\Omega_\tau} \subset \mathcal{K}\text{-}\underline{\lim}\, \Omega_t$. On the other hand, let $x \in \complement\overline{\Omega_\tau}$. As $(t,x) \in \complement\overline{Q}$ that is open, there exists a $\varepsilon > 0$ such that for $|t-\tau|$ small enough $d(x,\Omega_t) \geq \varepsilon$ and therefore $\mathcal{K}\text{-}\overline{\lim}\, \Omega_t \subset \overline{\Omega_\tau}$. As $\mathcal{K}\text{-}\underline{\lim}\, \Omega_t \subset \mathcal{K}\text{-}\overline{\lim}\, \Omega_t$, the proof is complete. ■

Proposition 41 *The sets Ω_0 and Ω_T are non-void.*

Proof – Let M_i be a connected component of M such that $M_i \cap ((0,T) \cap \mathbb{R}^n) \neq \varnothing$. Let $t_\downarrow$ and $t_\uparrow$ be some solutions of

$$t_\downarrow = \inf t,\ (t,x) \in M_i \text{ and } t_\uparrow = \sup t,\ (t,x) \in M_i$$

Let $x_\downarrow \in \mathbb{R}^n$ be such that $(t_\downarrow, x_\downarrow) \in \partial M_i$. Obviously at such a point $\nu_x = 0$. Because of the assumption (4.6), we have necessarily $t_\downarrow < 0$. By a similar argument, $t_\uparrow > T$. The connexity of M_i therefore yields that $M_i \cap (\{0\} \times \mathbb{R}^n)$ and $M_i \cap (\{T\} \times \mathbb{R}^n)$ are non-void. ■

We will sometimes use the notation

$$\Sigma = \overline{\partial Q \cap ((0,T) \times \mathbb{R}^n)} \tag{4.10}$$

Generation of the evolution sets by velocity fields. Let K be a smooth closed set of $\mathbb{R}^n$ sufficiently large (typically, $K = \overline{B(0,r)}$ for a big value of r). We consider the space of velocities

$$\mathcal{V} = \left\{V \in \mathcal{C}^0([0,T]; \mathcal{C}^1(K;\mathbb{R}^n)),\ \left\langle V, \nu^K \right\rangle = 0 \text{ on } \partial K\right\} \tag{4.11}$$

For any $V \in \mathcal{V}$, there is a unique solution $t \mapsto T_t \in \mathcal{C}^1([0,T]; \mathcal{C}^1(K;\mathbb{R}^n))$ of

$$\partial_t T_t = V(t) \circ T_t,\ t \in [0,T] \text{ and } T_0 = I \tag{4.12}$$

---

[1]Let K_n, $n \in \mathbb{N}$ be a sequence of subsets of $\mathbb{R}^n$. We recall that

$$\mathcal{K}\text{-}\overline{\lim}\, K_n = \{x \in \mathbb{R}^n,\ \liminf_{n\to+\infty} d(x,K_n) = 0\} \tag{4.7}$$

and

$$\mathcal{K}\text{-}\underline{\lim}\, K_n = \{x \in \mathbb{R}^n,\ \lim_{n\to+\infty} d(x,K_n) = 0\} \tag{4.8}$$

When $\mathcal{K}\text{-}\overline{\lim}\, K_n = \mathcal{K}\text{-}\underline{\lim}\, K_n$, this set is the *Kuratowski limit* of the K_n and is denoted $\mathcal{K}\text{-}\lim K_n$. These definition do still make sense when the previous limits $n \to +\infty$ are replaced with more general filters.

Proposition 42 *For any evolution set Q such that $\overline{Q} \subset [0,T] \times \mathrm{int}(K)$ there is a $V \in \mathcal{V}$ such that V "builds the tube Q"*, i.e.

$$\forall\, t \in [0,T],\ \Omega_t = T_t(\Omega_0) \tag{4.13}$$

Lemma 34 *For any $X_0 = (t_0, x_0) \in \Sigma$, there are some neighbourhoods $U_{\mathbb{R}}$, U_Γ and U of t_0 in $\mathbb{R}$, of X_0 in Γ_{t_0} and of X_0 in ∂M and a $\mathcal{C}^1$-diffeomorphism*

$$\varphi : \begin{array}{ccc} U_{\mathbb{R}} \times U_\Gamma & \to & U \\ (t,X) & \to & (t, \psi(t,X)) \end{array}$$

Proof – The set ∂M is a $\mathcal{C}^1$ n-submanifold of $\mathbb{R}^{n+1}$. For any $X_0 = (t_0, x_0)$ in Σ, there is a neighbourhood U of X_0 in ∂M and a $\mathcal{C}^1$ diffeomorphism $\phi = (\phi_t, \phi_x)$: $U \mapsto (t_0 - \Delta t, t_0 + \Delta t) \times (-1,1)^{n-1}$. As $\nu_x \neq 0$ for $t \in [0,T]$, $\{t\} \times \Gamma_t$ is a $(n-1)$-submanifold of ∂M and we can take ϕ such that $\phi_t(t,x) = t$. As the differential of ϕ is an isomorphism from $\mathrm{Tgt}(\{t\} \times \Gamma_t)$ to $\{0\} \times \mathbb{R}^{n-1}$, the differential of the mapping $(t,x) \in \Sigma \cap U \mapsto x' \in \Gamma_{t_0}$ such that $\phi_x(t,x) = \phi_x(0,x')$ is full rank. Let U_Γ be such that $U \cap (\{t_0\} \times \mathbb{R}^n) = \{t_0\} \times U_\Gamma$. We take for φ the inverse of the one-to-one mapping $(t,x) \in U \mapsto (t,x') \in (t_0 - \Delta t, t_0 + \Delta t) \times U_\Gamma$. ■

Now, we prove the proposition 42. By compacity of Σ, there is a finite number of such neighbourhoods $(U_i)_{i\in I}$ and associated mappings $(\varphi_i)_{i\in I}$ such that the set $A = \cup_{i\in I} U_i$ satisfies $\Sigma \subset A \subset \partial M$. Let $(O_j, \rho_j)_{j\in J}$ be a partition of the unity of A subordinated to $(U_i)_{i\in I}$. For any $(t,x) \in U_i$, we define $W_i(t,x)$ through

$$W_i(\varphi(t,X)) = \partial_t \varphi(t,X) \iff W_i(t, \psi(t,X)) = (1, \partial_t \psi(t,X))$$

Then we define $W : \Sigma \to \mathbb{R}^n$ by

$$W = \sum_{j\in J} \rho_j \cdot W_{i_j} \quad \text{where } i_j \in I \text{ is such that } O_j \subset U_{i_j}$$

It is of the form

$$W(t,x) = (1, V_\Sigma(t,x))$$

Let V be an extension of V_Σ in $\mathcal{C}^0([0,T]; \mathcal{C}^1(K; \mathbb{R}^n))$ such that the condition

$$\langle V, \nu \rangle = 0 \ \text{ on } \ [0,T] \times \partial K$$

is satisfied and let $t \mapsto T_t$ be the flow associated to V. Then the mapping $t \mapsto T_t$ builds Q. It is enough to prove that for any $t \in [0,T]$, we have $T_t(\Gamma_0) \subset \Gamma_t$. Therefore, as for any $x \in \Lambda_0$, $\partial_t(t, T_t(x)) = (1, V_\Sigma(t,x)) = W(t,x)$ and because W is a field of the tangent manifold to ∂M,

$$(t, T_t(x)) \in \partial M \cap (\{t\} \times \mathbb{R}^n) = \{t\} \times \Gamma_t$$

Remark 10 *We can see from the proof of the lemma 34 that the field V that generates Q can be taken in $\mathcal{C}^{k-1}([0,T]; \mathcal{C}^k(K; \mathbb{R}^n))$ whenever M is of class $\mathcal{C}^k$.*

Corollary 7 *The mapping*

$$\left(\begin{array}{ccc} [0,T] \times \Gamma_0 & \to & \Sigma \\ (t,x) & \mapsto & (t, T_t(x)) \end{array} \right)$$

is a $\mathcal{C}^1$ diffeomorphism.

Proof – We have $\Gamma_t = \partial\Omega_t = \partial T_t(\Omega_0) = T_t(\partial\Omega_0) = T_t(\Gamma_0)$ and therefore the mapping is one-to-one. The trigonal structure of $(\partial_t, D)(t, T_t(x))$ and the inversibility of $DT_t(x)$ yield the result. ■

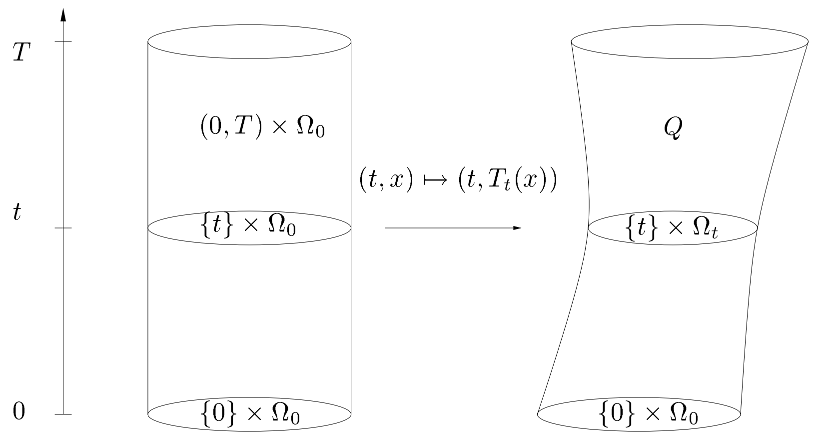


Figure 4.1: Generation of the non-cylindrical set Q.

4.1.2 Integration

Let $\nu = (\nu_t, \nu_x)$ be the pointing outward normal vector field to Σ. The pointing outward normal do Γ_t is denoted by n_t. As V builds Q, we have

$$n_t \propto {}^t[DT_t]^{-1} \cdot n$$

([2]) and in $(t, T_t(x)) \in \Sigma$, we have

$$\begin{aligned} \nu &\propto \left(\begin{bmatrix} 1 & 0 \\ \partial_t T_t & DT_t \end{bmatrix}^{-1} \right)^* \cdot \begin{pmatrix} 0 \\ n \end{pmatrix} \\ &= \begin{bmatrix} 1 & -\partial_t T_t^* \cdot {}^*[DT_t]^{-1} \\ 0 & {}^*[DT_t]^{-1} \end{bmatrix} \begin{pmatrix} 0 \\ n \end{pmatrix} \\ &\propto \begin{bmatrix} \langle -\partial_t T_t, n_t \rangle \\ n_t \end{bmatrix} \end{aligned}$$

Therefore, as $\partial_t T_t \circ T_t^{-1} = V$, we obtain in (t,x) the equalities

$$n_t = \frac{\nu_x}{\|\nu_x\|} \tag{4.14}$$

and

$$v = -\frac{\nu_t}{\|\nu_x\|} = \langle V, n_t \rangle \tag{4.15}$$

Proposition 43 *Assume that $g \in L^1(\Sigma; \mathcal{H}^n)$. Then,*

$$I_\Sigma(g) = \int_\Sigma g \, d\mathcal{H}^n = \int_0^T \left(\int_{\Gamma_t} g\sqrt{1+v^2}\, d\mathcal{H}^{n-1} \right) dt \tag{4.16}$$

[2]Remember that as $\Gamma_t = T_t(\Gamma_0)$, the tangent set to Γ_t in $T_t(x)$ is given by

$$\mathrm{Tgt}(\Gamma_t)(T_t(x)) = [DT_t(x)] \cdot \mathrm{Tgt}(\Gamma_0)(x)$$

Proof – Let $M(A) = (\det DA)\,{}^t[DA]^{-1}$ be the matrix of the cofactors of DA, n be the relative normal to Γ_0 and n_t to Γ_t. Let $\bar{T}_t : (t,x) \mapsto (t, T_t(x))$. Notice that

$$\frac{\|M(\bar{T}_t)\cdot(0,n)\|}{\|M(T_t)\cdot n\|} = \frac{\|(-{}^tV\,{}^t[DT_t]^{-1}\cdot n,\ {}^t[DT_t]^{-1}\cdot n)\|}{\|\,{}^t[DT_t]^{-1}\cdot n\|} = \frac{\|(-v,n_t)\|}{\|n_t\|} = \sqrt{1+v^2}$$

On one hand, we have

$$\int_{\Gamma_t} g\sqrt{1+v^2}\,d\mathcal{H}^{n-1} = \int_{\Gamma_0} g\circ(t,T_t(x))\,\sqrt{1+v^2}\cdot\|M(T_t)\cdot n\|_{\mathbb{R}^n}\,d\mathcal{H}^n$$

and on the other hand,

$$\int_\Sigma g\,d\mathcal{H}^n = \int_{[0,T]\times\Gamma_0} g\circ(t,T_t(x))\,\|M(\bar{T}_t)\cdot(0,n)\|_{\mathbb{R}^{n+1}}\,d\mathcal{H}^{n+1}$$

which concludes the proof. ■

4.2 Non-Cylindrical Navier-Stokes Equations

4.2.1 The Penalized Navier-Stokes Equations

Notations and Spaces. Let D be a smooth open bounded set of $\mathbb{R}^3$ ([3]). We define the spaces

$$\mathcal{V} = \{u \in \mathcal{D}(D;\mathbb{R}^3),\ \operatorname{div} u = 0\} \tag{4.17}$$

$$H = \{u \in L^2(D;\mathbb{R}^3),\ \operatorname{div} u = 0,\ \langle u,n\rangle = 0 \text{ on } \partial D\} \tag{4.18}$$

$$V = \{u \in H_0^1(D;\mathbb{R}^3),\ \operatorname{div} u = 0,\ u = 0 \text{ on } \partial D\} \tag{4.19}$$

The space $\mathcal{V}$ is dense in H and V. To simplify the latter calculations, we define the norm on V and V' by

$$\|u\|_V = \|\nabla u\|_{L^2} \text{ and therefore } \|f\|_{V'} = \sup_{\|u\|_V=1}\langle f,u\rangle \tag{4.20}$$

Thanks to the Poincaré inequality, this definition of the norm of V is equivalent to the H^1 norm. Let Q be an evolution set included in $(0,T)\times D$. In the sequel, we call *Homogeneous Penalized Navier-Stokes Equations* (HPNSE or simply PNSE) the system formally defined by

$$\begin{array}{cl} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \dfrac{1}{\varepsilon}\chi_{\complement Q}u + \nabla p = f & \text{in } D\times]0,T[\\ \operatorname{div} u = 0 & \text{in } D\times]0,T[\\ u = 0 & \text{on } \partial D\times]0,T[\\ u(0) = u_0 & \text{in } D \end{array} \tag{4.21}$$

Following for example [48] we define the solutions of that system by

Definition 18 *Let $u_0 \in H$ and $f \in L^2([0,T];V')$. A mapping u of $L^2([0,T];V)\cap L^\infty([0,T];H)$ is a* (weak) solution *of the system (4.21) iff for any $\phi \in \mathcal{C}^\infty([0,T];\mathbb{R})$ such that $\phi(T) = 0$, we have*

$$-u_0\phi(0) + \int_0^T [-u\cdot\dot{\phi} - \nu\Delta u\cdot\phi + (u\cdot\nabla)u\cdot\phi - \frac{1}{\varepsilon}\chi_{\complement Q}u - f]\,dt = 0 \ \textit{ in } V' \tag{4.22}$$

[3]In the sequel, D can always be taken under the form $D = B(0,R)$ for a $R > 0$ large enough.

4.2.2 The test function space issue

We present in the next section three classes of test function that may be used to characterize in a equivalent way the solutions of the NSE. The first class, constituted of separable test functions naturally comes from the defintion 18 and is frequently used in the NSE literature. However, there is no obvious counterpart of this set of test functions that may be used in the cylindrical case, contrary to the two other classes.

4.2.2.1 The Cylindrical Case

Let $u \in L^2([0,T];V) \cap L^\infty([0,T];H)$. We define the operator

$$N_\varepsilon(u) = -\left\langle u, \frac{\partial \cdot}{\partial t} \right\rangle + \nu \nabla u \cdot\cdot \nabla - (u \otimes u) \cdot\cdot \nabla + \frac{1}{\varepsilon} \left\langle \chi_{\complement Q} u, \cdot \right\rangle - f \tag{4.23}$$

Definition 19 *Let $u_0 \in V$ and $\varepsilon \in (0;+\infty]$. The function u is a solution of the PNSE with respect to the class $\mathcal{F}$ of test functions if*

$$\forall \psi \in \mathcal{F}, \int_{(0,T)\times D} N_\varepsilon(u) \cdot \psi \, dtdx = \int_D \langle u_0, \psi(0,\cdot) \rangle_{\mathbb{R}^3} \, dx \tag{4.24}$$

Definition 20 (Separable test functions) *The mapping ψ belongs to $\mathcal{F}_\otimes$ if it is of the form*

$$\psi(t,x) = \sum_{i \in I} \phi_i(t) \cdot \omega_i(x) \tag{4.25}$$

where I is finite, $\phi_i \in \mathcal{C}^\infty([0,T];\mathbb{R})$, $\phi_i(T) = 0$ and $\omega_i \in \mathcal{V}$.

This space of test functions comes naturally from the definition 18. Indeed, a mapping u is a weak solution of the PNSE in the sense of the definition 18 iff it is a solution with respect to the set $\mathcal{F}_\otimes$ of test functions : for any $\psi = \sum_{i \in I} \phi_i \otimes \omega_i$ with I, ϕ_i and ω_i as in the definition 20, the equation (4.22) holds with $\phi = \phi_i$. Testing the i-th equation on ω_i and summing the results, we obtain (4.24).

Definition 21 (Test functions of "minimal" regularity) *We set*

$$\begin{aligned} \mathcal{F}_m \quad = \quad & \{\psi \in L^2([0,T];V),\ \partial_t \psi \in L^2([0;T];L^2(D)), \\ & \nabla \psi \in L^2([0,T];L^3(D)),\ \psi(T,\cdot) \equiv 0\} \end{aligned} \tag{4.26}$$

We endow $\mathcal{F}_m$ with the norm

$$\|\psi\|_{\mathcal{F}_m} = \|\psi\|_{L^2(H^1)} + \|\partial_t \psi\|_{L^2(L^2)} + \|\nabla \psi\|_{L^2(L^3)} \tag{4.27}$$

Remark 11 *As any $\psi \in \mathcal{F}_m$ satisfies $\psi \in L^2([0,T];L^2(D))$ and $\partial_t \psi \in L^2([0,T];L^2(D))$, for any $t \in [0,T]$, $\psi(t,\cdot)$ does make sense (as a trace) in $L^2(D)$ for example (see [33]).*

The space $\mathcal{F}_m$ is important because it is a "large"[4] space on which the following property hold

Lemma 35 *The linear form $N_\varepsilon(u) : \mathcal{F}_m \to \mathbb{R}$ is continuous.*

[4]larger than the space of smooth functions $\mathcal{F}_\infty$ that is to be introduced. The corresponding topology is sufficiently weak to make easier the proof of some density results.

Proof – For any $\psi \in \mathcal{F}_m$, we have

$$\left| -\int \left\langle u, \frac{\partial \psi}{\partial t} \right\rangle dxdt \right| \leq \sqrt{T} \cdot \|u\|_{L^\infty(L^2)} \|\partial_t \psi\|_{L^2(L^2)}$$

$$\left| \int \nu \nabla u \cdot\cdot \nabla \psi \, dxdt \right| \leq \nu \cdot \|u\|_{L^2(V)} \|\nabla \psi\|_{L^2(V)}$$

Let K be such that $\|u\|_{L^6} \leq \|u\|_V$. We have

$$\begin{aligned} \left| \int (u \otimes u) \cdot\cdot \nabla \psi \, dxdt \right| &\leq \int_0^T \|u\|_{L^6} \|u\|_{L^2} \|\nabla \psi\|_{L^3} \, dt \\ &\leq K \cdot \|u\|_{L^2(V)} \|u\|_{L^\infty(L^2)} \|\nabla \psi\|_{L^2(L^3)} \end{aligned}$$

$$\left| \int \frac{1}{\varepsilon} \left\langle \chi_{\complement Q} u, \psi \right\rangle \, dxdt \right| \leq \lambda \|u\|_{L^\infty(L^2)} \|\psi\|_{L^2(V)}$$

for a suitable λ and of course

$$\langle f, \psi \rangle \leq \|f\|_{L^2(V')} \|\psi\|_{L^2(V)}$$

That concludes the proof. ■

Definition 22 (Smooth test functions) *We denote $\mathcal{F}_\infty$ the set of functions of $\mathcal{C}^\infty([0,T] \times \overline{D}; \mathbb{R}^3)$ whose support is included in $[0,T) \times D$ and such that $\operatorname{div} \psi = 0$.*

As stated in [35], we have

Lemma 36 *The set $\mathcal{F}_\infty$ is dense in $\mathcal{F}_m$.*

From separable to classical test functions. We prove the following result.

Proposition 44 *The set $\mathcal{F}_\otimes$ is dense in $\mathcal{F}_m$.*

We begin the proof by the construction of a projector on divergence-free fields : for a $u \in H_0^1(D)$, let Pu be the solution of

$$\|\nabla(u - Pu)\|_{L^2} = \inf_{v \in V} \|\nabla(u - v)\|_{L^2} \tag{4.28}$$

The operator P satisfies the following properties:

Lemma 37 *For any $p \in [2; +\infty)$, the operator $P : W_0^{1,p} \to W^{1,p} \cap V$ is linear continuous. Moreover there exists a $\mu > 0$ such that for any $u \in W_0^{1,p}$ such that if $\operatorname{div} u = 0$ we have*

$$\|Pu_n - u\|_{W^{1,p}} \leq \mu \|u_n - u\|_{W^{1,p}} \tag{4.29}$$

Proof – Standard calculations show that Pu is the solution v of the following Stokes problem

$$\begin{aligned} -\Delta v + \nabla \pi = -\Delta u &\quad \text{in } D \\ \operatorname{div} v = 0 &\quad \text{in } D \\ v = 0 &\quad \text{on } \partial D \end{aligned}$$

The classical properties of the Stokes system directly yield the continuity of P: indeed, there is a constant $K > 0$ such that

$$\|v\|_{W^{1,p}} \leq K \|\Delta u\|_{W^{-1,p}} \leq \lambda_p K \|u\|_{W^{1,p}}$$

(where λ_p is the norm of $\Delta : W^{1,p} \to W^{-1,p}$). If $u_n \to u$ in $W^{1,p}$ with $\operatorname{div} u = 0$, then $\delta_n = Pu_n - u$ is the solution of

$$\begin{aligned} -\Delta\delta_n + \nabla\pi_n &= -\Delta(u - u_n) && \text{in } D \\ \operatorname{div}\delta_n &= 0 && \text{in } D \\ \delta_n &= 0 && \text{on } \partial D \end{aligned}$$

and therefore

$$\|Pu_n - u\|_{W^{1,p}} = \|\delta_n\|_{W^{1,p}} \leq K\|\Delta(u - u_n)\|_{W^{-1,p}} \leq \lambda_p K\|u - u_n\|_{W^{1,p}}$$

■

Lemma 38 *Let $\psi \in \mathcal{F}_\infty$. There is a sequence ψ_n of mappings with*

$$\psi_n(t,x) = \sum_{i\in I_n} \phi_{i,n}(t) \cdot \omega_{i,n}(x) \tag{4.30}$$

where I_n is finite, $\phi_{i,n} \in \mathcal{C}^\infty([0,T];\mathbb{R})$, $\phi_{i,n}(T) = 0$ and $\omega_{i,n} \in \mathcal{D}(D)^3$ such that

$$\psi_n \to \psi \ \text{ in } \mathcal{F}_m \tag{4.31}$$

Proof – let $\eta^n : \mathbb{R}^n \to \mathbb{R}$ be the (n-dimensional) standard mollifier and for any $\varepsilon > 0$, $\eta^n_\varepsilon = (1/\varepsilon^n)\eta^n(\cdot/\varepsilon)$. We associate to ψ the mapping $\psi' : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\psi'(t,x) = \begin{cases} \psi(t,x) & \text{if } (t,x) \in [0,T]\times D \\ \psi(-t,x) & \text{if } (t,x) \in [-T,0]\times D \\ 0 & \text{in the other cases.} \end{cases}$$

Then, for a $\varepsilon > 0$, we define the mapping $\psi''_\varepsilon : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ by

$$\psi''_\varepsilon(t,x) = [(\tau \mapsto \psi'(\tau,x)) * \eta^1_\varepsilon](t)$$

There is a sequence $\varepsilon_n > 0$ such that

$$\|\psi''_{\varepsilon_n} - \psi\|_{\mathcal{F}_m} \leq \frac{1}{2n} \tag{4.32}$$

Now, we use Yosida's proposition (th. 1, p. 65, [49]) on each component of ψ''_{ε_n}. It yields the existence of some $\phi_{i,n} \in \mathcal{D}(\mathbb{R})$ and $\omega_{i,n} \in \mathcal{D}(\mathbb{R}^3)^3$ with

$$\operatorname{Supp}\phi_{i,n} \in (-\infty;T) \ \text{ and } \ \operatorname{Supp}\omega_{i,n} \subset D$$

and such that

$$\left\|\sum_{i\in I_n}\phi_{i,n}\otimes\omega_{i,n} - \psi''_{\varepsilon_n}\right\|_{\mathcal{F}_m} \leq \frac{1}{2n}$$

The two inequalities conclude the proof. ■

We finally prove the desired density result. Because of the density of $\mathcal{F}_\infty$ into $\mathcal{F}_m$, it is enough to prove that any $\psi \in \mathcal{F}_\infty$ may be approximated by a sequence of mappings of $\mathcal{F}_\otimes$. Let $\phi_{i,n}$ and $\omega_{i,n}$ be as in the previous lemma. Now, we define the function $\bar\psi_n : [0;T]\times D$ by

$$\bar\psi_n = \sum_{i\in I_n}\phi_{i,n}\otimes\bar\omega_{i,n} \ \text{ where } \ \bar\omega_{i,n} = P[\omega_{i,n}] * \eta^3_\varepsilon$$

where P is given by (4.28). As $\sum_{i\in I_n} \phi_{i,n} \otimes \omega_{i,n}$ converges in $L^2([0,T];W^{1,3})$ to the divergence-free field ψ, by the lemma 37, we have

$$P\left[\sum_{i\in I_n} \phi_{i,n} \otimes \omega_{i,n}\right] = \sum_{i\in I_n} \phi_{i,n} \otimes P[\omega_{i,n}] \to \psi \text{ in } L^2([0,T];W^{1,3})$$

For the same reason

$$\sum_{i\in I_n} \partial_t \phi_{i,n} \otimes P[\omega_{i,n}] \to \partial_t \psi \text{ in } L^2([0,T];L^2)$$

The spatial mollification is applied to ensure that $\bar{\omega}_{i,n} \in \mathcal{V}$. Suitable choices of the ε_n yield the convergence of $\bar{\psi}_n$ to ψ in $\mathcal{F}_m$.

As a consequence of these density results and the lemma 36, we have

Proposition 45 *The solutions u of the PNSE with respect to the classes $\mathcal{F}_\infty$, $\mathcal{F}_m$ and $\mathcal{F}_\infty$ are the same.*

4.2.2.2 The General (non-cylindrical) Case

Non-cylindrical spaces. Let Q be a $\mathcal{C}^2$ evolution set and let B_t be for any $t \in [0,T]$ a Banach space that contains mappings from Ω_t to $\mathbb{R}^m$. For any $p \in [1,+\infty]$, we set

$$L^p([0,T];B_t) = \left\{f : Q \to \mathbb{R}^m,\ f(t,\cdot) \in B_t \text{ a.e. in } [0,T],\ \|f\|_{L^p([0,T];B_t)} < +\infty\right\} \quad (4.33)$$

with, for $p < +\infty$

$$\|f\|^p_{L^p([0,T];B_t)} = \int_0^t \|f(t,\cdot)\|^p_{B_t}\, dt \quad (4.34)$$

and

$$\|f\|_{L^\infty([0,T];B_t)} = \sup_{t\in[0,T]} \|f(t,\cdot)\|_{B_t} \quad (4.35)$$

Test functions. We review shortly the facts corresponding to the previous section but in the non-cylindrical case. Let Q be a $\mathcal{C}^2$ evolution set and $V \in \mathcal{C}^1([0,T];\mathcal{C}^2(K;\mathbb{R}^n))$ a velocity field that generates Q. We set

$$\theta(t,x) = (t, T_t(x)) \quad (4.36)$$

Let ϕ_θ be the operator given by $\phi_\theta(\psi) := (\gamma_\theta^{-1} J_\theta \cdot \psi) \circ (t,\theta)^{-1}$ (see the section 4.3 for the study of this operator). We have already defined the set $\mathcal{F}_m$ for the cylindrical sets. Now, we define

$$\mathcal{F}_m(Q) = \phi_\theta(\mathcal{F}_m((0,T) \times \Omega_0)) \quad (4.37)$$

The operators ϕ_θ induce isomorphisms between $\mathcal{F}_m((0,T) \times \Omega_0)$ and $\mathcal{F}_m(Q)$ that may equivalently be described by

$$\begin{aligned}\mathcal{F}_m(Q) \quad = \quad &\{\psi \in L^2([0,T];V(\Omega_t)),\ \partial_t\psi \in L^2([0;T];L^2(\Omega_t)),\\ &\nabla\psi \in L^2([0,T];L^3(\Omega_t)),\ \psi(T,\cdot) \equiv 0\}\end{aligned} \quad (4.38)$$

We also define for $k \in \mathbb{N} \cup \{+\infty\}$ the space $\mathcal{F}_k(Q)$.

$$\mathcal{F}_k(Q) = \{\psi \in \mathcal{C}^k(\overline{Q};\mathbb{R}^3),\ \mathrm{Supp}\,\psi \subset (\{0\} \times \Omega_0) \cup Q,\ \mathrm{div}\,\psi = 0\} \quad (4.39)$$

Lemma 39 *The set $\mathcal{F}_\infty(Q)$ is dense in $\mathcal{F}_m(Q)$.*

Proof – We only prove that $\mathcal{F}_1(Q)$ is dense in $\mathcal{F}_m(Q)$. As $\phi_\theta^{-1}(\mathcal{F}_1(Q)) = \mathcal{F}_1((0,T)\times\Omega_0)$ and $\mathcal{F}_\infty((0,T)\times\Omega_0) \subset \mathcal{F}_1((0,T)\times\Omega_0) \subset \mathcal{F}_m((0,T)\times\Omega_0)$, $\phi_\theta^{-1}(\mathcal{F}_1(Q))$ is dense in $\mathcal{F}_m((0,T)\times\Omega_0)$. Therefore, $\mathcal{F}_1(Q)$ is dense in the space $\phi_\theta(\mathcal{F}_m((0,T)\times\Omega_0)) = \mathcal{F}_m(Q)$. ∎

4.2.3 Existence of a Penalized Solution

We state an existence result for the penalized system together with the estimates needed to show the existence of a non-cylindrical solution. The space involved in the 4th estimate is defined later in the proof (see definition 4.59 and lemma 42 for its compact inclusion properties).

Proposition 46 (Existence of a penalized solution) *We set*

$$E = \|u_0\|_{L^2}^2 + \frac{1}{\nu}\|f\|_{L^2(0,T;V')}^2 \tag{4.40}$$

For any $\varepsilon > 0$, there is a solution u_ε of the system (4.21). It can be chosen such that the following inequalities hold.

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{L^2}^2 \leq E \tag{4.41}$$

$$\|\nabla u_\varepsilon\|_{L^2(0,T;L^2)}^2 \leq \frac{E}{\nu} \tag{4.42}$$

$$\|\chi_{\complement Q}\cdot u_\varepsilon\|_{L^2(0,T;L^2)}^2 \leq \frac{\varepsilon E}{2} \tag{4.43}$$

Moreover, for any $\gamma < 1/4$, there is a $K > 0$ independent of ε such that

$$\|u_\varepsilon\|_{\mathcal{H}^\gamma([0,T])} \leq K \tag{4.44}$$

Proof – These results are obtained by an adaptation of the classical proof of the existence of a weak solution for the (cylindrical) NSE. For that reason, some parts of the proof are only sketched and we refer the reader to [48] or [33] for more details.

4.2.3.0.1 (i) Galerkin method. We consider a sequence $(v_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{V}$ which is free and total in V. For any $m \in \mathbb{N}$, we set

$$V_m = \mathrm{Span}(\{v_1, ..., v_m\}) \subset V \tag{4.45}$$

For a given $m \in \mathbb{N}$, we search for an approximate solution u_m of the penalized problem such that

$$\text{for any } \ t \in [0,T],\ u_m(t) \in V_m \tag{4.46}$$

This approximate solution is determined by the system

$$\int_D \langle \dot{u}_m, v\rangle\, dx + \nu \int_D \langle \nabla u_m, \nabla v\rangle\, dx + \int_D \langle (u_m\cdot\nabla)\cdot u_m, v\rangle\, dx \\ + \frac{1}{\varepsilon}\int_D \langle \chi_{\complement Q} u_m, v\rangle\, dx = \langle f, v\rangle_{V'\times V} \quad \text{for any } v \in V_m \tag{4.47}$$

or equivalently

$$\dot{u}_m - \nu \Delta u_m + (u_m \cdot \nabla) u_m + \frac{1}{\varepsilon} \chi_{\complement Q} u_m - f \in V_m^{\perp} \tag{4.48}$$

together with the initial condition

$$u_m(0) = P_m(u_0) \tag{4.49}$$

where P_m is the orthogonal projection in H on V_m. This system is equivalent to an ordinary differential equation (ODE) for the coefficients $\langle u_m(t), v_i \rangle$, $i \in \{1, ..., m\}$.

(ii) Energy and Penalization Estimates. We easily prove that for any order m, the approximate solution satifies

$$\frac{1}{2} \frac{d}{dt} \|u_m\|_{L^2}^2 + \frac{\nu}{2} \|\nabla u_m\|_{L^2}^2 + \frac{1}{\varepsilon} \|\chi_{\complement Q} \cdot u_m\|_{L^2}^2 \leq \frac{1}{2\nu} \|f\|_{V'}^2 \tag{4.50}$$

and therefore that with $E = \|u_0\|_{L^2}^2 + \frac{1}{\nu} \|f\|_{L^2(0,T;V')}^2$, we have

$$\begin{gathered} \sup_{t \in [0,T]} \|u_m(t)\|_{L^2}^2 \leq E, \ \|\nabla u_m\|_{L^2(0,T;L^2)}^2 \leq \frac{E}{\nu} \\ \text{and } \ \|\chi_{\complement Q} \cdot u_m\|_{L^2(0,T;L^2)}^2 \leq \frac{\varepsilon E}{2} \end{gathered} \tag{4.51}$$

(iii) Extra estimate and Compactness. To build by a limit process a solution of the penalized Navier-Stokes Equations, we need an extra estimate that is for example provided by the study of the fractional derivatives of some extensions of u_m.

By setting

$$F = f + \nu \Delta u_m - (u_m \cdot \nabla) u_m \tag{4.52}$$

we may rewrite the system (4.47) under the form $\langle \dot{u}_m, v \rangle_{L^2} = \left\langle F - \frac{1}{\varepsilon} \chi_{\complement Q} u_m, v \right\rangle_{V' \times V}$ or equivalently[5]

$$\langle \dot{\overline{u_m}}, v \rangle_{L^2} = \left\langle \overline{F} - \frac{1}{\varepsilon} \overline{\chi_{\complement Q} u_m}, v \right\rangle_{V' \times V} + \langle \overline{u_m}(0), v \rangle_{L^2} \delta_0 - \langle \overline{u_m}(T), v \rangle_{L^2} \delta_T$$

[5] Let $\bar{\cdot}$ be the extension by 0 outside $[0, T]$. The solution x of the system $\dot{x} = f(t)$ on $[0, T]$ satisfies for any $\phi \in \mathcal{D}(\mathbb{R})$

$$\int_{\mathbb{R}} \bar{\dot{x}}(t) \phi(t)\, dt = \int_{\mathbb{R}} \bar{f}(t) \phi(t)\, dt$$

By integration by parts, we obtain

$$- \int_{\mathbb{R}} \bar{x}(t) \dot{\phi}(t)\, dt = x(0)\phi(0) - x(T)\phi(T) + \int_{\mathbb{R}} \bar{f}(t) \phi(t)\, dt$$

or equivalently

$$\dot{\bar{x}} = \bar{f}(t) + x(0) \cdot \delta_0 - x(T) \cdot \delta_T$$

By Fourier transform [6] , we obtain

$$\forall\,\omega \in \mathbb{R},\ (i\omega)\,\langle \hat{u}_m(\omega), v\rangle_{L^2} \;=\; \left\langle \hat{F} - \frac{1}{\varepsilon}\widehat{\chi_{\complement Q} u_m}, v \right\rangle_{V'\times V} + \frac{1}{\sqrt{2\pi}}\left\langle u_m(0) - u_m(T)e^{-i\omega T}, v\right\rangle_{L^2} \tag{4.56}$$

We state two elementary lemmas to exploit this equation.

Lemma 40 *The function $\omega \mapsto \|\hat{F}(\omega)\|_{V'}$ is bounded*

Proof – a classical Fourier inequality is

$$\sup_{\omega\in\mathbb{R}} \|\hat{F}(\omega)\|_{V'} \leq \frac{1}{\sqrt{2\pi}}\|F\|_{L^1([0,T];V')} \tag{4.57}$$

The right-hand side f belongs to $L^2([0,T];V') \hookrightarrow L^1([0,T];V')$. On the other hand, the mapping $\psi : u \in V \mapsto \nu\Delta u - (u\cdot\nabla)u \in V'$ is continuous. As u_m belongs to a bounded set of $L^2([0,T];V)$ (see (4.51)), $\psi(u_m)$ belongs to a bounded set of $L^2([0,T];V) \hookrightarrow L^1([0,T];V)$ and therefore, $F = f + \psi(u_m)$ is bounded in $L^1([0,T];V')$. The inequality (4.57) gives the desired result. ■

We also notice that due to (4.51)

Lemma 41 *There is a $M > 0$ such that $\|u_m(0)\|_{L^2} \leq M$ and $\|u_m(T)\|_{L^2} \leq M$.*

Now, we set $v = \hat{u}_m(\omega)$ in (4.56). The Fourier transform being an isometry, we have

$$\left\langle \widehat{\chi_{\complement Q} u_m}, \hat{u}_m\right\rangle_{L^2} = \left\langle \chi_{\complement Q} u_m, u_m\right\rangle_{L^2} \geq 0$$

and therefore, using the lemmas 40 and 41, we obtain

$$|\omega|\|\hat{u}_m(\omega)\|_{L^2}^2 \leq K_1\|\hat{u}_m\|_V + K_2\|\hat{u}_m\|_{L^2}$$

for some constants K_1 and K_2. Finally, there is a $K > 0$ such that

$$|\omega|\|\hat{u}_m(\omega)\|_{L^2}^2 \leq K\cdot\|\nabla\hat{u}_m\|_{L^2}$$

At that point, we have have obtained the same kind of inequality that in the non-penalized case. The sequel follows therefore the proof of [48]. For any $\gamma > 1/2$, there is a $K_\lambda > 0$ such that

$$\forall\,\omega\in\mathbb{R},\ |\omega|^{2\lambda} \leq K_\lambda \frac{1+|\omega|}{1+|\omega|^{1-2\gamma}}$$

[6] Let H be a (complex) Hilbert space. For any $f \in L^1(\mathbb{R};H)\cap L^2(\mathbb{R};H)$, we set

$$[\mathcal{F}(f)](\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\cdot\exp(-i\omega t)\,dt \tag{4.53}$$

The operator $\mathcal{F}$ has a unique continuous extension to $L^2(\mathbb{R};H)$ and, still denoted $\mathcal{F}$. Moreover this operator is an isometry of $L^2(\mathbb{R};H)$. For any $f\in L^2(\mathbb{R};H)$ and any $\gamma > 0$ such that $|\cdot|^\gamma\mathcal{F}(f) \in L^2(\mathbb{R};H)$, we define $\partial^\gamma f \in L^2(\mathbb{R};H)$ by

$$\mathcal{F}(\partial^\gamma f) = |\omega|^\gamma \exp\left(i\gamma\frac{\pi}{2}\right)\cdot\mathcal{F}(f) \tag{4.54}$$

For any $f\in L^2([0,T];H)$, we also set

$$\bar{f}(t) = \begin{cases} f(t) & \text{if } t\in[0,T] \\ 0 & \text{if } t\notin[0,T]\end{cases} \quad \text{and} \quad \hat{f} = \mathcal{F}(\bar{f}) \tag{4.55}$$

Therefore

$$\int_{-\infty}^{+\infty} |\omega|^{2\gamma} \|\hat{u}_m(\omega)\|_{L^2}^2 \, d\omega \le K K_\lambda \int_{-\infty}^{+\infty} \frac{\|\nabla \hat{u}_m(\omega)\|_{L^2}}{1+|\omega|^{1-2\gamma}} \, d\omega + K_\lambda \int_{-\infty}^{+\infty} \|\hat{u}_m\|_{L^2}^2 \, d\omega$$

On one hand, the Parseval equality yields

$$\int_{-\infty}^{+\infty} \|\hat{u}_m\|_{L^2}^2 \, d\omega = \|u_m\|_{L^2([0,T];H)}^2 \le K' \text{ by } (4.51)$$

and on the other hand, for any $\gamma < 1/4$,

$$K'' = \int_{-\infty}^{+\infty} \frac{d\omega}{(1+|\omega|^{1-2\gamma})^2} < +\infty$$

and therefore

$$\int_{-\infty}^{+\infty} |\omega|^{2\gamma} \|\hat{u}_m(\omega)\|_{L^2}^2 \, d\omega \le K K_\lambda \sqrt{K''} \sqrt{\int_{-\infty}^{+\infty} \|\nabla u_m\|_{L^2}^2 \, dt} + K_\lambda K' \quad (4.58)$$

By (4.51), we already know that the sequence of extensions $\bar{u}_m$ of u_m is bounded in $L^2(\mathbb{R}; V)$, therefore, it is bounded in the set $\mathcal{H}^\gamma([0,T])$, defined by

$$\mathcal{H}^\gamma(K) = \{f \in L^2(\mathbb{R}; V),\ |\cdot|^\gamma \mathcal{F}(f) \in L^2(\mathbb{R}; H),\ \mathrm{Supp}\, f \subset K\} \quad (4.59)$$

(iv) Construction of the solution. We have proved that the sequence u_m of approximate solutions of the system (4.47) are bounded in the spaces $L^2([0,T]; V)$, $L^\infty([0,T]; H)$ (see (4.51)) and that $\bar{u}_m$ is bounded in $\mathcal{H}^\gamma([0,T])$ (see (4.58)). It is shown in [48] that

Lemma 42 *The space $\mathcal{H}^\gamma([0,T])$ is compactly included in $L^2(\mathbb{R}; H)$.*

Therefore, there exists a subsequence of u_m and a $u \in L^2([0,T]; V) \cap L^\infty([0,T]; H)$ such that

$$\begin{gathered} u_m \rightharpoonup u \text{ in } L^2([0,T]; V), \quad u_m \overset{\star}{\rightharpoonup} u \text{ in } L^\infty([0,T]; H) \\ \text{and } u_m \to u \text{ in } L^2([0,T]; H) \end{gathered} \quad (4.60)$$

Lemma 43 *Let the family u_λ satisfy*

$$u_\lambda \to u \ \text{ strongly in } \ L^2([0,T]; H)$$

Then for any $\psi \in L^\infty([0,T]; W^{1,\infty}(D; \mathbb{R}^3))$, we have

$$\int_0^T \langle (u_\lambda \cdot \nabla) u_\lambda, \psi \rangle_{V' \times V} \, dt \to \int_0^T \langle (u \cdot \nabla) u, \psi \rangle_{V' \times V} \, dt \ \text{ when } \ \lambda \to 0 \quad (4.61)$$

Proof – Because $\mathrm{div}\, u_\lambda = 0$ and $u_\lambda|_{\partial D} = 0$, we have

$$\langle (u_\lambda \cdot \nabla) u_\lambda, \psi \rangle_{V' \times V} = \langle \mathrm{div}(u_\lambda \otimes u_\lambda), \psi \rangle_{V' \times V} = - \langle u_\lambda \otimes u_\lambda, D\psi \rangle_{V' \times V}$$

and therefore,

$$\int_0^T \langle (u_\lambda \cdot \nabla) u_\lambda, \psi \rangle_{V' \times V} \, dt = - \int_0^T \langle u_\lambda \otimes u_\lambda, D\psi \rangle_{V' \times V} \, dt$$

The strong convergence $u_\lambda \to u$ in $L^2([0,T];H)$ implies the strong convergence $u_\lambda \otimes u_\lambda \to u \otimes u$ in $L^1([0,T];L^1(D;\mathbb{R}^{3\times 3}))$. It yields the desired convergence. ■

A multiplication by a test function followed by an integration by parts in the system (4.107) yields that for any $\phi \in \mathcal{C}^\infty([0,T];\mathbb{R})$ such that $\phi(T)=0$, we have

$$-u_m(0)\phi(0) + \int_0^T [-u_m \cdot \dot{\phi} - \nu\Delta u_m \cdot \phi + (u_m \cdot \nabla)u_m \cdot \phi - \frac{1}{\varepsilon}\chi_{\complement Q} u_m - f]\, dt \in V_m^\perp \tag{4.62}$$

Let $m \in \mathbb{N}$, $v \in V_m$ and $n \geq m$. As $u_n(0) = P_n(u_0)$, $u_n(0) \to u_0$ in H and

$$\langle u_n(0), \phi(0)v\rangle_{V'\times V} \to 0\, \langle u(0), \phi(0)v\rangle_{V'\times V} \text{ when } n \to +\infty \tag{4.63}$$

As $u_n \rightharpoonup u$ in $L^2([0,T];H)$, we have

$$\int_0^T -\langle\, u_n, \dot{\phi v}\,\rangle_{V'\times V} + \left\langle -1/\varepsilon \cdot \chi_{\complement Q} u_n - f, \phi v\right\rangle_{V'\times V}\, dt \longrightarrow$$
$$\int_0^T -\langle\, u, \dot{\phi v}\,\rangle_{V'\times V} + \left\langle -1/\varepsilon \cdot \chi_{\complement Q} u - f, \phi v\right\rangle_{V'\times V}\, dt \tag{4.64}$$

The convergence $u_n \rightharpoonup u$ in $L^2([0,T];V)$ yields

$$\nu \int_0^T \langle \nabla u_n, \phi \nabla v\rangle_{L^2}\, dt \to \nu \int_0^T \langle \nabla u, \phi\nabla v\rangle_{L^2}\, dt \tag{4.65}$$

and $u_n \to u$ in $L^2([0,T];H)$ gives, by the lemma 43

$$\int_0^T \langle (u_n \cdot \nabla)u_n, \phi v\rangle_{V'\times V}\, dt \to \int_0^T \langle (u\cdot\nabla)u, \phi v\rangle_{V'\times V}\, dt \tag{4.66}$$

Finally, for any $m \in \mathbb{N}$

$$-u_0\phi(0) + \int_0^T [-u \cdot \dot{\phi} - \nu\Delta u \cdot \phi + (u\cdot\nabla)u \cdot \phi - \frac{1}{\varepsilon}\chi_{\complement Q} u - f]\, dt \in V_m^\perp \tag{4.67}$$

and therefore, as $\bigcup_{m\in\mathbb{N}} V_m = V$, u is solution of

$$-u_0\phi(0) + \int_0^T [-u \cdot \dot{\phi} - \nu\Delta u \cdot \phi + (u\cdot\nabla)u \cdot \phi - \frac{1}{\varepsilon}\chi_{\complement Q} u - f]\, dt = 0 \text{ in } V' \tag{4.68}$$

that is, of the desired penalized problem. ■

4.2.4 Existence of a Non-Cylindrical Solution

Proposition 47 *Let $u_0 \in H(\Omega_0)$ and $f \in L^2([0;T];V'(\Omega_t))$. There is a solution $u \in L^2([0,T];V(\Omega_t)) \cap L^\infty([0,T];L^2(\Omega_t))$ of the system*

$$\begin{array}{ll} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } Q \\ \operatorname{div} u = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u_0 & \text{in } \Omega_0 \end{array} \tag{4.69}$$

in the class $\mathcal{F}_\infty(Q)$ of test functions

Proof – We set

$$\overline{u}_0(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega_0 \\ 0 & \text{if } x \in \complement\Omega_0 \end{cases} \tag{4.70}$$

Obviously $\operatorname{div}\overline{u}_0 = 0$ in Ω_0 and in $\operatorname{int}\complement\Omega_0$. As $u_0|_{\Gamma_0} = 0$, we have $\operatorname{div}\overline{u}_0 = 0$ in $\mathbb{R}^3$ and therefore $\overline{u}_0 \in H(D)$. Let $\bar{f} \in L^2([0,T]; V'(D))$ be an extension of f. Let u_ε be a solution of the system

$$\begin{array}{cc} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \dfrac{1}{\varepsilon}\chi_{\complement Q} + \nabla p = \bar{f} & \text{in } D\times]0,T[\\ \operatorname{div} u = 0 & \text{in } D\times]0,T[\\ u = 0 & \text{on } \partial D\times]0,T[\\ u(0) = \bar{u}_0 & \text{in } D \end{array} \tag{4.71}$$

satsfying the estimates (4.41) to (4.44). Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^4;\mathbb{R}^3)$ such that $\operatorname{Supp}\psi \subset (\{0\}\times\Omega_0)\cup Q$ and $\operatorname{div}\psi = 0$. As $\chi_{\complement Q} = 0$ a.e. in $\operatorname{Supp}\psi$, we have for any $\varepsilon > 0$

$$-\langle u_0, \psi(0,\cdot)\rangle_{L^2} + \int_0^T [-\langle u_\varepsilon, \dot{\psi}\rangle_{L^2} + \nu\langle\nabla u_\varepsilon, \nabla\psi\rangle_{L^2} + \langle (u_\varepsilon\cdot\nabla)u_\varepsilon - f, \psi\rangle_{V'\times V}]\, dt = 0 \tag{4.72}$$

From the estimates (4.41) to (4.44) and the compactness lemma 42, we can extract a subsequence of the u_ε such that

$$\begin{array}{c} u_\varepsilon \rightharpoonup u \text{ in } L^2([0,T];V), \quad u_\varepsilon \overset{\star}{\rightharpoonup} u \text{ in } L^\infty([0,T];H) \\ \text{and } u_\varepsilon \to u \text{ in } L^2([0,T];H) \end{array} \tag{4.73}$$

With the help of the lemma 43, it comes that u is solution of

$$-\langle u_0, \psi(0,\cdot)\rangle_{L^2} + \int_0^T [-\langle u, \dot{\psi}\rangle_{L^2} + \nu\langle\nabla u, \nabla\psi\rangle_{L^2} + \langle (u\cdot\nabla)u - f, \psi\rangle_{V'\times V}]\, dt = 0 \tag{4.74}$$

Moreover, as $\|\chi_{\complement Q} u_\varepsilon\|_{L^2([0,T];L^2)} \le K\varepsilon$ (see (4.43)), $\|\chi_{\complement Q} u\|_{L^2([0,T];L^2)} = 0$ and therefore $u = 0$ almost everywhere on $\complement Q$. As a consequence, u belongs to $L^2([0,T]; H(\Omega_t))$. ■

4.3 Transformation Calculus

Let Q be a $\mathcal{C}^2$ evolution set. Let $\operatorname{Diff}_2(\overline{Q})$ be the set of $\mathcal{C}^2$ mappings $\theta : \overline{Q} \to \mathbb{R}^3$ such that on $\theta(0,x) = x$ for any $x \in \Omega_0$ and for every $(t,x) \in \overline{Q}$, $D_x\theta(t,x)$ is invertible. For any such mapping $\theta : \overline{Q} \to \mathbb{R}^3$, we set

$$Q_\theta = \{(t, \theta(t,x)) \in \mathbb{R}\times\mathbb{R}^3,\ (t,x) \in Q\} \tag{4.75}$$

We also denote t the mapping $(t,x) \to t$. Then, we have equivalently

$$Q_\theta = (t,\theta)(Q) \tag{4.76}$$

By abuse of notation, $\theta^{-1} : Q_\theta \to \mathbb{R}\times\mathbb{R}^3$ is the mapping such that

$$(t, \theta^{-1}) = (t,\theta)^{-1} \tag{4.77}$$

We set

$$J_\theta = D\theta \ \text{ and } \ \gamma_\theta = \det D\theta \tag{4.78}$$

Obviously

$$\det D(t,\theta) = \gamma_\theta \tag{4.79}$$

By the above convention, we have

Lemma 44 *Let $\theta \in \mathrm{Diff}_2(\overline{Q})$. We have*

$$D(\theta^{-1}) = J_\theta^{-1} \circ (t, \theta^{-1}) \text{ and } \partial_t(\theta^{-1}) = -[J_\theta^{-1} \cdot \partial_t \theta] \circ (t, \theta^{-1}) \tag{4.80}$$

Proof – By definition of θ^{-1}, for any $(t,x) \in Q$, $\theta^{-1}(t, \theta(t,x)) = x$. The differentiation with respect to x gives the first part of (4.80). The differentiation with respect to t gives $\partial_t(\theta^{-1})(t,\theta(x)) + [D(\theta^{-1})(t,\theta(t,x))] \cdot \partial_t\theta(t,x) = 0$. That proves the second part of (4.80). ■

Transport of functions. At that point, we define the operators ϕ_θ. For any $u \in L^2(Q;\mathbb{R}^3)$, we set

$$\phi_\theta(u) := (\gamma_\theta^{-1} J_\theta \cdot u) \circ (t,\theta)^{-1} \in L^2(Q_\theta;\mathbb{R}^3) \tag{4.81}$$

and the conjugate transformations ψ_θ by

$$\psi_\theta = [\phi_\theta^*]^{-1} \tag{4.82}$$

Lemma 45 *The explicit expression of ψ_θ is given by*

$$\psi_\theta(u) = ([J_\theta^*]^{-1} \cdot u) \circ (t,\theta)^{-1} \tag{4.83}$$

Proof – For any $\theta \in \mathrm{Diff}_2(\overline{Q})$, $u \in L^2(Q_\theta;\mathbb{R}^3)$ and $v \in L^2(Q;\mathbb{R}^3)$

$$\langle \psi_\theta^{-1}(u), v\rangle = \langle u, \phi_\theta(v)\rangle = \int_{Q_\theta} \langle u, (\gamma_\theta^{-1} J_\theta \cdot v) \circ (t,\theta)^{-1}\rangle \, dx$$

As $\det D(t,\theta^{-1}) = \det [D(t,\theta)^{-1}] = \gamma_\theta^{-1} \circ (t,\theta)^{-1}$, it yields

$$\begin{aligned}
\langle \psi_\theta^{-1}(u), v\rangle &= \int_{Q_\theta} \langle u \circ (t,\theta), J_\theta \cdot v\rangle \circ (t,\theta)^{-1} \cdot \gamma_\theta^{-1} \circ (t,\theta)^{-1} \, dx \\
&= \int_Q \langle u \circ (t,\theta), J_\theta \cdot v\rangle \, dx = \langle J_\theta^* \cdot u \circ (t,\theta), v\rangle
\end{aligned}$$

and finally, $\psi_\theta^{-1}(u) = J_\theta^* \cdot u \circ (t,\theta)$. That proves the equation (4.83). ■

The operators ϕ_θ also induce some isomorphisms between $L^2([0,T];H_0^1(\Omega_t))$ and $L^2([0,T];H_0^1(\Omega_{\theta,t}))$ (with $\{t\} \times \Omega_{\theta,t} = Q_\theta \cap (\{t\} \times \mathbb{R}^3)$). As a consequence of the following corollary, they also induce some isomorphisms between $L^2([0,T];V(\Omega_t))$ and $L^2([0,T];V(\Omega_{\theta,t}))$. This extends the domain of definition of the operators ψ_θ to $L^2([0,T];H^{-1}(\Omega_t))$ and to $L^2([0,T];V'(\Omega_t))$.

Corollary 8 *Let $\theta \in \mathrm{Diff}_2(\overline{Q})$, $u \in L^2([0,T];H_0^1(\Omega_t))$, $v \in L^2([0,T];L^2(\Omega_{\theta,t}))$ and $\pi \in L^2(Q_\theta)$. We have*

$$\psi_\theta^{-1}(\nabla\pi) = \nabla(\pi \circ (t,\theta)) \tag{4.84}$$

$$\langle \mathrm{div}\, \phi_\theta(u), v\rangle = \langle \mathrm{div}\, u, v \circ (t,\theta)\rangle \tag{4.85}$$

Proof – The first equation is a direct consequence of $\psi_\theta^{-1}(\nabla\pi) = J_\theta^* \cdot (\nabla\pi) \circ (t,\theta)$. The second is obtained by duality : $\langle \operatorname{div} \phi_\theta(u), v\rangle = -\left\langle u, \psi_\theta^{-1}(\nabla v)\right\rangle = -\langle u, \nabla(v \circ (t,\theta))\rangle = \langle \operatorname{div} u, v\circ(t,\theta)\rangle$. ■

Lemma 46 *Let $\theta \in \mathrm{Diff}_2(\overline{Q})$. On the subset of $L^2(Q)$ formed by the fonctions u such that $\partial_t u \in L^2(Q)$, we have*

$$\psi_\theta^{-1} \circ \frac{\partial}{\partial t} \circ \phi_\theta = M_\theta \cdot \frac{\partial}{\partial t} + K_\theta^1 \cdot\cdot D + L_\theta^1 \tag{4.86}$$

with

$$M_\theta = \gamma_\theta^{-1} J_\theta^* J_\theta \ , \ L_\theta^1 = -J_\theta^*[J_\theta \cdot \partial_t\theta]_i \cdot \partial_i(\gamma_\theta^{-1} J_\theta) \tag{4.87}$$

$$K_\theta^1 = \gamma_\theta^{-1} J_\theta^* J_\theta \otimes J_\theta^{-1} \partial_t\theta \tag{4.88}$$

Proof – Let

$$I = \left\langle \psi_\theta^{-1} \circ \frac{\partial}{\partial t} \circ \phi_\theta(u), \phi \right\rangle = \left\langle \partial_t[(\gamma_\theta^{-1} J_\theta \cdot u) \circ (t,\theta^{-1})], (\gamma_\theta^{-1} J_\theta \cdot \phi) \circ (t,\theta^{-1})\right\rangle$$

We have

$$\begin{aligned} \partial_t[(\gamma_\theta^{-1} J_\theta \cdot u)\circ(t,\theta^{-1})] &= (\partial_t(\gamma_\theta^{-1} J_\theta)\cdot u)\circ(t,\theta^{-1}) \\ &\quad +(\gamma_\theta^{-1} J_\theta \cdot \partial_t u)\circ(t,\theta^{-1}) \\ &\quad [D(\gamma_\theta^{-1} J_\theta \cdot u)\circ(t,\theta^{-1})]\cdot\partial_t(\theta^{-1}) \end{aligned}$$

and

$$\begin{aligned} [D(\gamma_\theta^{-1} J_\theta \cdot u)\circ(t,\theta^{-1})]\cdot\partial_t(\theta^{-1}) &= [(\gamma_\theta^{-1} J_\theta Du)\circ(t,\theta^{-1})]\cdot\partial_t(\theta^{-1}) \\ &\quad +[\partial_t(\theta^{-1})]_i \cdot [\partial_i(\gamma_\theta^{-1} J_\theta)\cdot u]\circ(t,\theta^{-1}) \\ &= -(\gamma_\theta^{-1} J_\theta Du J_\theta^{-1}\cdot\partial_t\theta)\circ(t,\theta^{-1}) \\ &\quad -([J_\theta\cdot\partial_t\theta]_i\cdot\partial_i(\gamma_\theta^{-1} J_\theta)\cdot u)\circ(t,\theta^{-1}) \end{aligned}$$

With M_θ, L_θ^1 given in (4.87) and K_θ^1 in (4.88), we have

$$\begin{aligned} I &= \int_{Q_\theta} \left\langle M_\theta\cdot\partial_t u + K_\theta^1 \cdot\cdot Du + L_\theta^1 \cdot u, \phi\right\rangle \circ (t,\theta^{-1})\cdot\gamma_\theta^{-1}\circ(t,\theta^{-1})\, dx \\ &= \int_Q \left\langle M_\theta\cdot\partial_t u + K_\theta^1 \cdot\cdot Du + L_\theta^1\cdot u, \phi\right\rangle\, dx \end{aligned}$$

■

Lemma 47 *Let $\theta \in \mathrm{Diff}_2(\overline{Q})$. On the set $L^2([0,T]; V(\Omega_t))$, we have*

$$\psi_\theta^{-1} \circ \Delta \circ \phi_\theta = \operatorname{div}(A_\theta D(\cdot)) + K_\theta^2 \cdot\cdot D + L_\theta^2 \tag{4.89}$$

with

$$A_\theta = \gamma_\theta^{-1} J_\theta^* J_\theta \tag{4.90}$$

$$[L_\theta^2]_{i,j} = \sum_{\delta,\nu}\left[-\gamma_\theta\cdot\left\langle\nabla(\gamma_\theta^{-1}[J_\theta]_{\delta,i}), \nabla(\gamma_\theta^{-1}[J_\theta]_{\delta,j})\right\rangle + \partial_\nu([J_\theta]_{\delta,i}\partial_\nu(\gamma_\theta^{-1}[J_\theta]_{\delta,j}]))\right] \tag{4.91}$$

$$[K_\theta^2]_{i,j,k} = -\sum_\delta\left[\partial_k(\gamma_\theta^{-1}[J_\theta]_{\delta,i})[J_\theta]_{\delta,j} + [J_\theta]_{\delta,i}\partial_k(\gamma_\theta^{-1}[J_\theta]_{\delta,j})\right] \tag{4.92}$$

Proof – We set $I = \left\langle \psi_\theta^{-1} \circ \Delta \circ \phi_\theta(u), \phi \right\rangle$. We have

$$I = -\left\langle D[(\gamma_\theta^{-1} J_\theta \cdot u) \circ (t, \theta^{-1})], D[(\gamma_\theta^{-1} J_\theta \cdot \phi) \circ (t, \theta^{-1})] \right\rangle$$

Clearly

$$D[(\gamma_\theta^{-1} J_\theta \cdot u) \circ (t, \theta^{-1})] = [(\gamma_\theta^{-1} J_\theta Du) + [\partial_j(\gamma_\theta^{-1} [J_\theta]_{i,\cdot}) \cdot u]_{i,j}] \circ (t, \theta^{-1})$$

and the analogous formula holds for $D[(\gamma_\theta^{-1} J_\theta \cdot \phi) \circ (t, \theta^{-1})]$. Then

$$I = -\langle A_\theta Du, D\phi \rangle + \left\langle L_\theta^2 \cdot u, \phi \right\rangle + \left\langle K_\theta^2 \cdot\cdot Du, \phi \right\rangle$$

■

Lemma 48 *Let* $\theta \in \mathrm{Diff}_2(\overline{Q})$*. On the set* $L^\infty([0,T]; V(\Omega_t)) \cap L^2([0,T]; H^2(\Omega_t))$*, we have*

$$\psi_\theta^{-1} \circ [u \mapsto -(u \otimes u) \cdot\cdot D] \circ \phi_\theta = N_\theta^* D[N_\theta u]u \;\; \textit{with} \;\; N_\theta = \gamma_\theta^{-1} J_\theta \tag{4.93}$$

Proof – Let $\phi \in \mathcal{D}(Q; \mathbb{R}^3)$ and let

$$I = \left\langle -\psi_\theta^{-1}((\phi_\theta(u) \otimes \phi_\theta(u)) \cdot\cdot D), \phi \right\rangle$$

We have

$$I = -\int_{Q_\theta} [(\gamma_\theta^{-1} J_\theta \cdot u) \otimes (\gamma_\theta^{-1} J_\theta \cdot u)] \circ (t, \theta)^{-1} \cdot\cdot D((\gamma_\theta^{-1} J_\theta \cdot \phi) \circ (t, \theta)^{-1})\, dxdt$$

As

$$D((\gamma_\theta^{-1} J_\theta \cdot \phi) \circ (t, \theta)^{-1}) = [D(\gamma_\theta^{-1} J_\theta \cdot \phi) \cdot J_\theta^{-1}] \circ (t, \theta)^{-1}$$

we obtain

$$\begin{aligned} I &= -\int_{Q_\theta} [[(\gamma_\theta^{-1} J_\theta \cdot u) \otimes (\gamma_\theta^{-1} J_\theta \cdot u)] \cdot\cdot [D(\gamma_\theta^{-1} J_\theta \cdot \phi) \cdot J_\theta^{-1}]] \circ (t, \theta)^{-1}\, dxdt \\ &= -\int_Q [\gamma_\theta^{-1} (J_\theta \cdot u) \otimes (J_\theta \cdot u)] \cdot\cdot [D(\gamma_\theta^{-1} J_\theta \cdot \phi) \cdot J_\theta^{-1}]\, dxdt \end{aligned}$$

Using the identity $A \cdot\cdot (BC) = (AC^*) \cdot\cdot B$, we come to

$$\begin{aligned} I &= -\int_Q [\gamma_\theta^{-1} (J_\theta u) \otimes u] \cdot\cdot [D(\gamma_\theta^{-1} J_\theta \phi)]\, dxdt \\ &= \int_Q \langle \mathrm{div}(\gamma_\theta^{-1} (J_\theta u) \otimes u), (\gamma_\theta^{-1} J_\theta \phi) \rangle\, dxdt \\ &= \int_Q \langle D(\gamma_\theta^{-1} J_\theta u) \cdot u, (\gamma_\theta^{-1} J_\theta \phi) \rangle\, dxdt \end{aligned}$$

■

4.4 Shape sensitivity of the NSE

We distinguish in the sequel the following degree of regularity of the solutions of the NSE.

Definition 23 (Regularity of the solutions) *A **weak solution** (or simply solution) belongs to* $L^2([0,T];V(\Omega_t))\cap L^\infty([0,T];L^2(\Omega_t))$.
*A **strong solution** belongs to* $L^2([0,T];H^2(\Omega_t))^3\cap L^\infty([0,T];V(\Omega_t))$.
*A **smooth solution** belongs to* $\mathcal{C}^\infty([0,T]\times\overline{Q})$.

To derive some shape sensitivity results for the solutions of the Navier-Stokes Equations, we essentially make a regularity assumption on the solution of the NSE *in the reference geometry*. We will not focus in the sequel on the conditions that allows such regularity because there exists an extensive literature on this major and still partially open subject *in the cylindrical case*. Roughly speaking, these results state that sufficient smoothness of the data yield the *short-time* smoothness of the solution u. Additional smallness assumptions on the data yield the global smoothness of the solution.

Most of the techniques involved in the proves of these regularity result have their non-cylindrical counterpart with obviously an extra technical layer. In the following situation, the smoothness assumption on u will clearly be satisfied:
• The data (f, Ω, u_0) in the reference set are smooth and compatible[7].
• The reference set is $Q=(0,T^*)\times\Omega$ (it is cylindrical) and T^* is small enough.

Theorem 6 *Let the mapping* $\theta\in\mathrm{Diff}_2(\overline{Q})\mapsto f_\theta\circ(t,\theta)\in L^2([0,T];L^2(Q))$ *be continuously differentiable. If u is a smooth solution of the NSE in Q, there is a neighbourhood ω of the identity in* $\mathrm{Diff}_2(\overline{Q})$ *and a unique shape-dependent mapping u such that*
(i) For $\theta=I$, $u_\theta=u$.
(ii) For any $\theta\in\omega$, u_θ is a solution of the NSE with data f_θ in Ω_θ.
(iii) The mapping $\theta\in\omega\mapsto u_\theta\circ(t,\theta)$ is continuously differentiable in the spaces $L^2([0,T];H^2(\Omega_t))$ *and* $L^\infty([0,T];H^1(\Omega_t))$.

The proof relies on the application of the implicit function theorem to the equations
$$\mathfrak{N}^\theta(u)=0,\ u(0)=u_0$$
where the transported Navier-Stokes operator is given by
$$\mathfrak{N}^\theta=\psi_\theta^{-1}\circ\mathfrak{N}_\theta\circ\phi_\theta \tag{4.94}$$
and $\mathfrak{N}_\theta$ is the Navier-Stokes operator in Ω_θ
$$\mathfrak{N}_\theta(u)=-\left\langle u,\frac{\partial\,\cdot}{\partial t}\right\rangle+\nu\nabla u\cdot\cdot\nabla-(u\otimes u)\cdot\cdot\nabla-f_\theta \tag{4.95}$$
the explicit expression of this operator being given by
$$\mathfrak{N}^\theta(u)=M_\theta\frac{\partial u}{\partial t}-\nu\cdot\mathrm{div}(A_\theta Du)+K_\theta\cdot\cdot Du+L_\theta\cdot u+N_\theta^* D[N_\theta u]u-\psi_\theta^{-1}(f_\theta) \tag{4.96}$$

[7] see [48]

with

$$K_\theta = K_\theta^1 + K_\theta^2 \text{ and } L_\theta = L_\theta^1 + L_\theta^2 \tag{4.97}$$

We prove the properties required to apply the IFT in the two following subsections.

4.4.1 Regularity of the differential.

Proposition 48 *Let Q be a $\mathcal{C}^2$ evolution set. The mapping*

$$(\theta, u) \in \mathrm{Diff}_2(\overline{Q}) \times \mathcal{A} \mapsto (\mathfrak{N}^\theta(u), u(0)) \in \mathcal{B} \tag{4.98}$$

where $\mathcal{A}$ is the set of functions u that belongs to $L^2([0,T]; H^2(\Omega_t))$, $L^\infty([0,T]; V(\Omega_t))$, such that $\frac{\partial u}{\partial t} \in L^2([0,T]; L^2(\Omega_t))$ and $\mathcal{B} = [L^2([0,T]; \pi(L^2(\Omega_t)))] \times V(\Omega_0)$. is continuously differentiable.

Proof – First, we check that for any $u \in \mathcal{A}$ and any admissible θ, $\mathfrak{N}^\theta(u)$ belongs to $\mathcal{B}$. It is clear that

$$f_1 = M_\theta \frac{\partial u}{\partial t} - \nu \cdot \mathrm{div}(A_\theta D u) + K_\theta \cdot\cdot D u + L_\theta \cdot u$$

belongs to $L^2([0,T]; L^2(\Omega_t))$. Moreover, for any $u \in \mathcal{A}$, $N_\theta u \in L^\infty([0,T]; V(\Omega_t))$ and therefore $D[N_\theta u] \in L^\infty([0,T]; L^2(\Omega_t))$. As $u \in L^2([0,T]; H^2(\Omega_t))$ that is continuously included in $L^2([0,T]; L^\infty(\Omega_t))$, we obtain that

$$f_2 = N_\theta^* D[N_\theta u] u \in L^2([0,T]; L^2(\Omega_t))$$

and therefore, $\mathfrak{N}^\theta(u) = \pi(f_1 + f_2)$ belongs to $L^2([0,T]; L^2(\Omega_t))$. As f_1 is linear continuous with respect to u and f_2 is bilinear continuous with respect to u, the corresponding partial derivative exists and is given by

$$\begin{aligned} \partial_u \mathfrak{N}^\theta(u) \cdot (\delta u) &= M_\theta \frac{\partial(\delta u)}{\partial t} - \nu \cdot \mathrm{div}(A_\theta D(\delta u)) + K_\theta \cdot\cdot D(\delta u) && (4.99)\\ &+ L_\theta \cdot (\delta u) + N_\theta D[J_\theta(\delta u)] u + N_\theta D[J_\theta u](\delta u) && (4.100) \end{aligned}$$

and the regularity of θ yields the existence of $\partial_\theta \mathfrak{N}^\theta$ whose expression is

$$\begin{aligned} \partial_\theta \mathfrak{N}^\theta(u) \cdot (\delta\theta) &= (\partial_\theta M_\theta \cdot (\delta\theta)) \frac{\partial u}{\partial t} - \nu \cdot \mathrm{div}((\partial_\theta A_\theta \cdot (\delta\theta)) D u) && (4.101)\\ &+ (\partial_\theta K_\theta \cdot (\delta\theta)) \cdot\cdot D u (\partial_\theta L_\theta \cdot (\delta\theta)) \cdot u + \cdots && (4.102)\\ &(\partial_\theta N_\theta \cdot (\delta\theta)) D[J_\theta u] u + N_\theta D[(\partial_\theta J_\theta \cdot (\delta\theta)) u] u && (4.103) \end{aligned}$$

■

4.4.2 Isomorphism property.

A- Uniqueness

Proposition 49 *Let u be a smooth*[8] *solution of the NSE on $[0,T]$ and δu a strong*

[8]The smoothness assumption on u is clearly suboptimal: the reader can check for example that the existence of a bound M such that

$$\int_0^T \|Du\|_{L^\infty}\, dt \le M$$

that is, $u \in L^1([0,T]; W^{1,\infty}(\Omega_t))$, yields the same result.

solution of

$$\frac{\partial(\delta u)}{\partial t} - \nu \cdot \Delta(\delta u) + [D(\delta u)] \cdot u + [Du] \cdot (\delta u) = 0 \ \textit{with} \ \delta(0) = 0 \tag{4.104}$$

Then, for any $t \in [0, T]$*, we have*

$$\frac{1}{2}\partial_t \|\delta u\|_{L^2}^2 + \nu \|D(\delta u)\|_{L^2}^2 \leq \left| \int_{\Omega_t} (\delta u)^* [Du](\delta u)\, dx \right| \tag{4.105}$$

and consequently $\delta u \equiv 0$*.*

Proof – The mapping δ being a strong solution of the linearized system, for any $\phi \in \mathcal{D}((0,T))$, the mapping $(t,x) \mapsto \phi(t)u(t,x)$ belongs to $\mathcal{F}_m$. Therefore, we have

$$\begin{aligned} -\int_Q \langle \delta u, \delta u\rangle\, \partial_t \phi\, dtdx - \int_Q \langle u, \partial_t u\rangle\, \phi\, dtdx + \cdots \\ \nu \int_Q \langle \nabla \delta u, \nabla \delta u\rangle\, \phi\, dtdx + \int_Q (\delta u)^*[Du](\delta u)\phi\, dtdx &= 0 \end{aligned}$$

We do not distinguish the mapping u and its extension by 0 outside Q. The previous inequality is equivalent in $\mathcal{D}'((0,T))$ to

$$\partial_t \|\delta u\|_{L^2(D)}^2 - \langle \delta u, \partial_t \delta u\rangle_{L^2(D)} + \nu \|\nabla \delta u\|_{L^2(D)}^2 + \int_D (\delta u)^*[Du](\delta u)\, dx = 0$$

That yields (4.105). On the other hand, the smoothness of u yields the existence of a $K > 0$ such that

$$\left| \int (\delta u)^* [Du](\delta u)\, dx \right| \leq K \|\delta u\|_{L^2}^2$$

Combined with (4.105), this equation yields $\|\delta u(t)\|_{L^2} \leq \|\delta(0)\|_{L^2} \exp(Kt/2)$ and therefore, $\delta u \equiv 0$. ■

B- Existence.

Proposition 50 *Let* u *be a smooth solution of the NSE in* $[0,T]$*. For any* $f \in L^2([0,T]; L^2)$ *and* $\delta u_0 \in V(\Omega_0)$*, there is a solution* δu *of the system*

$$\frac{\partial(\delta u)}{\partial t} - \nu \cdot \Delta(\delta u) + [D(\delta u)] \cdot u + [Du] \cdot (\delta u) = f \ \textit{in} \ V' \tag{4.106}$$

such that $\delta u(0) = \delta u_0$ *and* δu *belongs to the set* $\mathcal{A}$*.*

Proof – We go back to a Galerkin approximations scheme. We take the same notations that in the section 4.2.3.0.1. Let $\delta u_m : [0,T] \mapsto V_m$ be such that

$$\partial_t(\delta u_m) - \nu \Delta(\delta u_m) + (u \cdot \nabla)\delta u_m + (\delta u_m \cdot \nabla) u + \frac{1}{\varepsilon}\chi_{\complement Q}(\delta u_m) - f \in V_m^{\perp} \tag{4.107}$$

where u is a solution of the NSE, extended by 0 outside Q. We add the initial condition

$$\delta u_m(0) = P_m(\delta u_0) \tag{4.108}$$

Let $K = \|Du\|_{L^\infty}$. By calculations similar to the ones that led to the estimate (4.50), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\delta u_m\|_{L^2}^2 - K\|\delta u_m\|_{L^2}^2 + \frac{\nu}{2}\|\nabla\delta u_m\|_{L^2}^2 + \frac{1}{\varepsilon}\|\chi_{\complement Q}\cdot\delta u_m\|_{L^2}^2 \leq \frac{1}{2\nu}\|f\|_{V'}^2 \quad (4.109)$$

Therefore, the variable $\kappa = \delta u_m \exp(Kt)$ satisfies

$$\frac{1}{2}\frac{d}{dt}\|\kappa\|_{L^2}^2 \leq \left[-\frac{\nu}{2}\|\nabla\kappa\|_{L^2}^2 - \frac{1}{\varepsilon}\|\chi_{\complement Q}\cdot\kappa\|_{L^2}^2\right] + \frac{1}{2\nu}\|f\|_{V'}^2\exp(2Kt) \qquad (4.110)$$

That yields the uniform boundedness of δu_m in $L^2([0,T];V)$ and $L^\infty([0,T];L^2)$. Now, we use the test function $\partial_t(\delta u_m)$ and obtain

$$\begin{aligned}\|\partial_t\delta u_m\|_{L^2}^2 + \frac{\nu}{2}\partial_t\|\nabla\delta u_m\|_{L^2}^2 + \frac{1}{2\varepsilon}\partial_t\|\chi_{\complement Q}\partial_t(\delta u_m)\|_{L^2}^2 \leq \\ \langle f - (u\cdot\nabla)\delta u_m - (\delta u_m\cdot\nabla)u, \partial_t(\delta u_m)\rangle\end{aligned}$$

As $F = f - (u\cdot\nabla)\delta u_m - (\delta u_m\cdot\nabla)u$ belongs to a bounded set of $L^2([0,T];V')$ we have, with $M(t) = \|F(t)\|_{L^2}$

$$\frac{1}{2}\|\partial_t\delta u_m\|_{L^2}^2 + \frac{\nu}{2}\partial_t\|\nabla\delta u_m\|_{L^2}^2 + \frac{1}{2\varepsilon}\partial_t\|\chi_{\complement Q}(\delta u_m)\|_{L^2}^2 \leq \frac{1}{2}M(t)^2$$

and therefore, δu_m is bounded in $L^\infty([0,T];V)$ and $\partial_t\delta u_m$ in $L^2([0,T];L^2)$. Passing weakly to the limit in m, we obtain the existence of a solution δu of (4.106) such that

$$\delta u \in L^\infty([0,T],L^2) \ \text{ and } \ \partial_t\delta u \in L^2([0,T];L^2)$$

The mapping $\delta u(t) \in V(\Omega_t)$ is solution of

$$-\nu\Delta\delta u(t) = F(t) \ \text{ in } V'(\Omega_t) \ \text{ with } \ F = -\partial_t\delta u - [D(\delta u)]\cdot u - [Du]\cdot(\delta u) - f$$

There exists of a uniform bound K in t such that for any $t \in [0,T]$,

$$\|\delta u(t)\|_{H^2} \leq K\|F(t)\|_{L^2}$$

■

Appendix A

Basic facts about $\mathcal{C}^k$ spaces

A.1 $\mathcal{C}^k$ spaces on compact stable sets

In this section E denotes a topological space.

Definition 24 *A subset K of E is stable if $\mathrm{int}(K)$ is dense in K.*

Proposition 51 *Let U and K be two sets of E. Assume moreover that U is open and that $\overline{U} = K$. Then the following properties are equivalent:*

i) U is relatively compact[1].

ii) K is a compact stable set.

Proof –

i) $\Rightarrow$ ii) : Compactness is obvious. As $U \subset \mathrm{int}(\overline{U})$, density follows.

ii) $\Rightarrow$ i) : Clear. ■

In particular, when $E = \mathbb{R}^n$, the compact stable sets are exactly the closure of the open bounded sets: on one hand, a bounded open set U of $\mathbb{R}^n$ is automatically relatively compact (and therefore $K = \overline{U}$ is compact and stable) and on the other hand, for any such set K, $U = \mathrm{int}(K)$ is (one of) the corresponding open bounded set.

We have the following examples:

Example 1 The sets $[a, b] \subset \mathbb{R}$ (with $a < b$) are compact stable sets. The compact Lipschitz submanifolds of $\mathbb{R}^n$ of dimension N also are.

It is well-known that any continuous application $f : K \subset E \to F$, with E, F metric spaces, K compact, is uniformly continuous on K and therefore on any dense subset U of K (this is Heine's theorem). The following lemma gives a partial converse.

Lemma 49 *Let E and F be two metric spaces, F being complete, and U be a subset of E. Any uniformly continuous mapping $f : U \subset E \to F$ has a unique continuous extension $\bar{f}$ in $\overline{U}$. Moreover, it is uniformly continuous.*

[1] A subset of a topological space is relatively compact if its closure is compact.

Proof – Consider $x^* \in \overline{U}$ and a sequence $x_n \in U$ which converges to x^*. As x_n is a Cauchy sequence, $f(x_n)$ also is and converges to a $f^* \in F$. For any other sequence y_n which also converges to x^*, $d(x_n, y_n) \to 0$, so $d(f(x_n), f(y_n)) \to 0$, which implies that $f(y_n) \to f^*$. The element $\bar{f}(x^*) = f^*$ is consequently uniquely determined and obviously satisfies $\bar{f}(x) = f(x)$ when $x \in U$.

Moreover, the mapping $\bar{f}$ is uniformly continuous on $\overline{U}$: for any $\varepsilon > 0$, let $\eta > 0$ be such that

$$(x, y) \in U^2 \text{ and } d(x, y) < 2\eta \text{ imply } d(f(x), f(y)) < \varepsilon$$

Let $(x, y) \in \overline{U}^2$ be such that $d(x, y) < \eta$ and consider two sequences x_n and y_n of U which converge respectively to x and y. For n big enough, $d(x_n, y_n) < 2\eta$ and therefore $d(f(x_n), f(y_n)) \leq \varepsilon$. Passing to the limit, we obtain $d(\bar{f}(x), \bar{f}(y)) \leq \varepsilon$. ■

Assume now that E and F are metric spaces, the latter being complete, that K is a compact stable set of E and that U is an open set such that $\overline{U} = K$.

Then, if $f : U \to F$ is uniformly continuous, it has a unique continuous extension $g = \bar{f}$ on K. On the other hand, if $g : K \to F$ is continuous, and therefore uniformly continuous, its restriction f to U is uniformly continuous. Consequently, we may perform the identification

$$\mathcal{C}^0(\overline{U}; F) = \Big\{ f : U \to F,\ f \text{ uniformly continuous} \Big\} \tag{A.1}$$

This space, is a complete metric space when it is endowed with the distance of uniform convergence ; it is a Banach space when F is a Banach space, what we will assume in the sequel. This (re-)definition is especially useful as the function is not *a priori* required to be defined on the boundary. For $E = \mathbb{R}^n$, and for any $k \in \mathbb{N} \cup \{+\infty\}$, we may set

$$\mathcal{C}^k(\overline{U}; F) = \Big\{ f \in \mathcal{C}^k(U; F) \,|\, \forall\, \alpha,\ |\,\alpha| \leq k,\ \partial_x^\alpha f \in C^0(\overline{U}; F) \Big\} \tag{A.2}$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ is a multi-index, $|\alpha| = \alpha_1 + ... + \alpha_n$ and

$$\partial_x^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \tag{A.3}$$

From now on, we adopt the following abuse of notation:

Notation 2 We identify any function $f \in \mathcal{C}^k(\overline{U}; F)$ with its extension $\bar{f}$. Consequently, for any multi-integer α with $|\alpha| \leq k$, as $\partial_x^\alpha f \in \mathcal{C}^{k-|\alpha|}(\overline{U}; F)$, the term $\partial_x^\alpha f(x)$ does make sense for any $x \in \overline{U}$.

One could be object that the directional derivatives could be used to define in a simpler way the values of $\partial_x^\alpha f$ on the boundary. However, as it is shown in the following example, this latter method cannot be applied in general for dimensions greater than one.

Example 2 Let U be the unit ball of $\mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be the function defined by

$$f(x, y) = x$$

As for any $h \neq 0$, $(h,1) \notin \overline{U}$, we cannot define the value of $\partial_x f(1,0)$ directly by the formula

$$\partial_x f(a,b) = \lim_{h\to 0} \frac{\bar{f}(a+h,b) - \bar{f}(a,b)}{h}$$

However, it is clear that $f \in \mathcal{C}^\infty(\overline{U};\mathbb{R})$ and that $\partial_x f(x,y) = 1$ for any $(x,y) \in U$. Identifying $\partial_x f$ and its extension on $\overline{U}$, we obtain $\partial_x f(0,1) = 1$.

A.2 Multi-variable $\mathcal{C}^k$ functions

This subsection is dedicated to the connections between the regularity of a two-variable mapping

$$f \left(\begin{array}{ccc} K_1 \times K_2 & \to & F \\ (x,y) & \mapsto & f(x,y) \end{array} \right) \tag{A.4}$$

and the associated functional-valued mappings

$$f_x \left(\begin{array}{ccc} K_1 & \to & F^{K_2} \\ x & \mapsto & f(x,\cdot) \end{array} \right) \text{ and } f_y \left(\begin{array}{ccc} K_2 & \to & F^{K_1} \\ y & \mapsto & f(\cdot,y) \end{array} \right) \tag{A.5}$$

Throughout this section, K_1 and K_2 are compact and stable subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and F is a Banach space.

Lemma 50 *The following properties are equivalent:*

i) $f \in \mathcal{C}^0(K_1 \times K_2; F)$
ii) $f_x \in \mathcal{C}^0(K_1; \mathcal{C}^0(K_2; F))$
iii) $f_y \in \mathcal{C}^0(K_2; \mathcal{C}^0(K_1; F))$

Proof – As the variables x and y play symmetric roles in property *i)*, it is sufficient to show that *i)* and *ii)* are equivalent to prove the whole result.

i) $\Rightarrow$ *ii)* : Assume that f is in $\mathcal{C}^0(K_1 \times K_2; F)$. Then, for any $x \in K_1$, $f_x(x) \in \mathcal{C}^0(K_2; F)$. To show that the correspondance $x \mapsto f_x(x)$ is continuous, we have to show that for any $x \in K_1$, $\|f_x(x') - f_x(x)\|_{\mathcal{C}^0(K_2)} \to 0$ when $x' \to x$. If it was not true, by compacity we could find a $\varepsilon > 0$, some sequences x'_n in K_1, y_n in K_2, and a $y \in K_2$ such that $x'_n \to x,\ y_n \to y$ and nevertheless

$$\forall n \in \mathbb{N},\ \|f(x'_n, y_n) - f(x, y_n)\| \geq \varepsilon$$

That would contradict the uniform continuity of f at (x,y).

ii) $\Rightarrow$ *i)* : Assume now that $f_x \in \mathcal{C}^0(K_1; \mathcal{C}^0(K_2))$. For any $(x,y) \in K_1 \times K_2$ and any $(x',y') \in K_1 \times K_2$, we have

$$\|f(x',y') - f(x,y)\|_F \ \leq\ \|f(x,y') - f(x,y)\|_F + \|f_x(x') - f_x(x)\|_{\mathcal{C}^0(K_2;F)}$$

and therefore f is continuous in (x,y). ■

Proposition 52 *For any integers k and l, the following properties are equivalent:*

i) For any $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^m$ such that $|\alpha| \leq k$ and $|\beta| \leq l$, $\partial_x^\alpha \partial_y^\beta f$ exists and is in $\mathcal{C}^0(K_1 \times K_2; F)$.
ii) $f_x \in \mathcal{C}^k(K_1; \mathcal{C}^l(K_2; F))$
iii) $f_y \in \mathcal{C}^l(K_2; \mathcal{C}^k(K_1; F))$

When one of these properties is satisfied, for any multi-integers α and β such that $|\alpha| \leq k$ and $|\beta| \leq l$ and any $(x, y) \in K_1 \times K_2$, we have

$$\partial_x^\alpha \partial_y^\beta f(x,y) = \partial_y^\beta \left[\partial_x^\alpha f_x(x)\right](y) = \partial_x^\alpha \left[\partial_y^\beta f_y(y)\right](x) \tag{A.6}$$

Remark 12 In the point *i)* of the preceding proposition, the order in which the partial differentiation of f are made does not count. We show that property in the following simple case: f is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$ such that $\partial_x f$, $\partial_y f$ and $\partial_x \partial_y f$ exists and are continuous.

We first recall a classical result of the distribution theory. Let g be a continuous function whose partial derivative $\partial_x g$ exists and is continuous. Then, for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$, the integration by parts

$$\int_{\mathbb{R}^2} \frac{\partial g}{\partial y} \cdot \varphi \, dxdy = -\int_{\mathbb{R}^2} g \cdot \frac{\partial \varphi}{\partial y} \, dxdy$$

is legitimate: the weak derivative in the variable x of g, denoted $\widetilde{\partial}_x g$, is equal to the classical one. Conversely, the continuity of g and $\widetilde{\partial}_x g$ yields the existence of $\partial_x g$ and the same equality.

Under the preceding continuity assumptions on f, we have $\partial_x \partial_y f = \widetilde{\partial}_x \partial_y f = \widetilde{\partial}_x \widetilde{\partial}_y f$ and as the weak partial derivatives do permute, $\partial_x \partial_y f = \widetilde{\partial}_y \widetilde{\partial}_x f = \widetilde{\partial}_y \partial_x f$. Now, as $\partial_x f$ and $\widetilde{\partial}_y \partial_x f$ are continuous, we finally obtain the existence of $\partial_y \partial_x f$ and the desired equality $\partial_y \partial_x f = \partial_x \partial_y f$.

Proof – Again a symmetry argument, consequence of the preceding remark, makes the equivalence between *i)* and *ii)* and the first part of the equation (A.6) the only thing to prove.

i) $\Rightarrow$ *ii)* : Assume that $f_x \in \mathcal{C}^k(K_1; \mathcal{C}^l(K_2; F))$. For any $y \in \operatorname{int}(K_2)$, consider the linear continous operator $l(y)$:

$$l(y) \left(\begin{array}{ccc} \mathcal{C}^0(K_2; F) & \rightarrow & F \\ g & \mapsto & g(y) \end{array} \right)$$

For any $y \in \operatorname{int}(K_2)$, we have

$$f_y(y) = [x \mapsto f(x,y)] = l(y) \cdot f_x$$

The existence of $\partial_x^\alpha f_x(x)$ for any $|\alpha| \leq k$ and any $x \in \operatorname{int}(K_1)$ yields the existence of $\partial_x^\alpha f(x,y)$ and the equality

$$\partial_x^\alpha f(x,y) = \left[l(y) \cdot \partial_x^\alpha f_x\right](x) = [\partial_x^\alpha f_x(x)](y) \tag{A.7}$$

For any α, $|\alpha| \leq k$, $\partial_x^\alpha f_x \in \mathcal{C}^0(K_1; \mathcal{C}^j(K_2; F))$, so for any $|\beta| \leq l$, $\partial_y^\beta \left[\partial_x^\alpha f_x(x)\right]$ exists, belongs to $\mathcal{C}^0(K_2; F)$ and

$$x \mapsto \partial_y^\beta \left[\partial_x^\alpha f_x(x)\right] \in \mathcal{C}^0(K_1, \mathcal{C}^0(K_2; F))$$

Consequently, $\partial_y^\beta \partial_x^\alpha f(x,y)$ exists, is equal to $\partial_y^\beta \left[\partial_x^\alpha f_x(x)\right](y)$ (see equation (A.7)) and the mapping $\partial_y^\beta \partial_x^\alpha f \in \mathcal{C}^0(K_1 \times K_2; F)$ (consequence of the lemma 50).

i) ⇒ ii) : We assume that for any multi-integer α, β such that $|\alpha| \leq k$ and $|\beta| \leq l$, $\partial_y^\beta \partial_x^\alpha f$ exists and is in $\mathcal{C}^0(K_1 \times K_2; F)$. We consider the following property:

$$\mathcal{P}(i,j) \left| \begin{array}{l} \text{The mapping } f_x \text{ belongs to } \mathcal{C}^i(K_1;\mathcal{C}^j(K_2;F)). \text{ Moreover, for} \\ \text{any } |\alpha| \leq i, |\beta| \leq j \text{ and } (x,y) \in \mathrm{int}(K_1) \times \mathrm{int}(K_2), \text{ the equality} \\ \partial_y^\beta[\partial_x^\alpha f_x(x)](y) = \partial_x^\beta \partial_y^\alpha f(x,y) \text{ holds.} \end{array} \right.$$

It is shown to be true by induction for any couple $i \leq k$ and $j \leq l$. Notice that if the property $\mathcal{P}(i,j)$ is satisfied, Lemma 50 implies that $x \mapsto \partial_y^\beta[\partial_x^\alpha f_x(x)] \in \mathcal{C}^0(K_1;\mathcal{C}^0(K_2))$ for any α and β of order smaller than i and j.

• Step 0: $i = 0,\ j = 0$. This is a direct consequence of the lemma 50.

• Step 1: $i \leq k,\ j = 0$. We assume that $\mathcal{P}(i',0)$ is satisfied for any $i' \leq i-1$. Let α be a multi-integer such that $|\alpha| = i$. We consider $\alpha' \in \mathbb{N}^n$ and $t \in \{1,...,n\}$ such that $\alpha = \alpha' + e_t$. For any $x \in \mathrm{int}(K_1)$, we have

$$\begin{aligned} &\left\| \frac{\partial_x^{\alpha'} f_x(x+he_t) - \partial_x^{\alpha'} f_x(x)}{h} - [\partial_x^\alpha f]_x(x) \right\|_{\mathcal{C}^0(K_2)} \\ &\qquad = \left\| \frac{[\partial_x^{\alpha'} f]_x(x+he_k) - [\partial_x^{\alpha'} f]_x(x)}{h} - [\partial_x^\alpha f]_x(x) \right\|_{\mathcal{C}^0(K_2)} \\ &\qquad \leq \sup_{\lambda \in [0,1]} \left\| [\partial_x^\alpha f]_x(x+\lambda he_t) - [\partial_x^\alpha f]_x(x) \right\|_{\mathcal{C}^0(K_2)} \end{aligned}$$

As $\partial_x^\alpha f \in \mathcal{C}^0(K_1 \times K_2; F)$, $[\partial_x^\alpha f]_x \in \mathcal{C}^0(K_1;\mathcal{C}^0(K_2;F))$ and the right-hand side converges to zero when $h \to 0$, which shows that $[\partial_x^\alpha f_x](x)$ exists and is equal to $[\partial_x^\alpha f]_x(x)$ and the property $\mathcal{P}(i,0)$ is proved.

• Step 2 : $i \leq k,\ j \leq l$. We use the same notations that in the previous step. Let β be a multi-integer of order smaller than j. We have

$$\begin{aligned} &\left\| \partial_x^\beta \left[\frac{\partial_x^{\alpha'} f_x(x+he_t) - \partial_x^{\alpha'} f_x(x)}{h} - [\partial_x^\alpha f]_x(x) \right] \right\|_{\mathcal{C}^0(K_2)} \\ &\qquad = \left\| \frac{\partial_y^\beta \partial_x^{\alpha'} f(x,y+he_k) - \partial_y^\beta \partial_x^{\alpha'} f(x,y)}{h} - \partial_y^\beta \partial_x^\alpha f(x,y) \right\|_{\mathcal{C}^0(K_2)} \\ &\qquad \leq \sup_{\lambda \in [0,1]} \left\| \partial_y^\beta \partial_x^\alpha f(x,y+\lambda he_k) - \partial_y^\beta \partial_x^\alpha f(x,y) \right\|_{\mathcal{C}^0(K_2)} \end{aligned}$$

and this term converges to zero as $\partial_y^\beta \partial_x^\alpha f$ is continous at (x,y). So, $\partial_y^\beta[\partial_x^\alpha f_x(x)](y)$ exists and is equal to $\partial_y^\beta \partial_x^\alpha f(x,y)$: $\mathcal{P}(i,j)$ is satisfied. ■

A.3 Sensitivity in the ODEs

A.3.1 Prerequisite

In this section, K is a compact set of $\mathbb{R}^n$ such that $\overline{D} \subset K$ and $T > 0$. We define the set $\mathcal{H} = \{f \in \mathcal{C}^0(\mathbb{R}^n;\mathbb{R}^n),\ \mathrm{Supp}(f) \subset K\}$ and denote $\|\cdot\|_k$ the usual norm on the set $\mathcal{C}^k(\overline{A};F)$. If F is itself of the form $F = \mathcal{C}^l(\overline{B};G)$, we also use the notation $\|\cdot\|_{k,l}$.

Lemma 51 *The mappings*

$$\begin{array}{ccc} \mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}))^2 & \to & \mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R})) \\ (f,g) & \mapsto & fg \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C}^0([-T;T];\mathcal{H}) \times \mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}^n)) & \to & \mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}^n)) \\ (f,g) & \mapsto & [t \mapsto f(t)\circ g(t)] \end{array}$$

are continuous.

Proof –
i) clear (remember that $\mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R})) \simeq \mathcal{C}^0([-T;T]\times\overline{D};\mathbb{R})$).
ii) for any f, f' in $\mathcal{C}^0([-T;T];\mathcal{H})$, and g, g' in $\mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}^n))$, we have

$$\|f'\circ g' - f\circ g\|_{0,0} \le \|f'-f\|_{0,0} + \|f\circ g' - f\circ g\|_{0,0}$$

It is therefore enough to show the uniform equicontinuity of the family $(f(t))_{t\in[-T;T]}$. It is proved as follows: For any $t \in [-T;T]$, and any $\varepsilon > 0$, there is a $\eta > 0$ such that $\|x-y\| < \eta$ implies $\|f(t)(x) - f(t)(y)\| < \varepsilon/3$ (by uniform continuity of $f(t)$). Let $\tau > 0$ be such that for any $t' \in [-T;T]\cap]t-\tau;t+\tau[$, $\|f(t') - f(t)\|_0 < \varepsilon/3$. Then, for any x and y such that $\|x-y\| < \eta$, $\|f(t')(x) - f(t')(y)\| < \varepsilon$. By compactness of $[-T;T]$, the result is proved. ■

Assume that D is of class $\mathcal{C}^2$. Using an extension operator, we may identify the velocity space $\mathcal{V}_2$ with a linear subspace of the set of continuous mappings from $[-T;T]$ to $\{f \in \mathcal{C}^2(\mathbb{R}^n;\mathbb{R}^n),\ \mathrm{Supp}(f) \subset K\}$. For any mapping $\xi \in \mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and $v \in \mathcal{V}_2$, we set

$$H(\xi,v) = [t \mapsto v(t)\circ\xi(t)] \tag{A.8}$$

and we state the

Lemma 52 *The expression (A.8) defines a $\mathcal{C}^1$ mapping*

$$H : \mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n)) \times \mathcal{V}_2 \to \mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$$

whose differential is given by:

$$\begin{aligned} DH(\xi,v)\cdot(\zeta,W) &= \partial_\xi H(\xi,v)\cdot\zeta + \partial_v H(\xi,v)\cdot W && \text{(A.9)} \\ &= \Big[t \mapsto [D_x v(t)\circ\xi(t)]\,\zeta(t) + W(t)\circ\xi(t)\Big] && \text{(A.10)} \end{aligned}$$

Proof – The mapping H is well-defined: indeed for any $t \in [-T;T]$, ξ belongs to $\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and $v \in \mathcal{V}_2$, the function $H(\xi,v)(t)$ is differentiable and $D_x[H(\xi,v)(t)] = [D_x v(t)]\circ\xi(t)\cdot D_x\xi(t)$. The lemma 51 yields that $D_x H(\xi,v)$ belongs to $\mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}^{n\times n}))$ and therefore $H(\xi,v) \in \mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^{n\times n}))$.

• Existence and continuity of $\partial_\xi H$: let $\mathcal{A}_{\xi,v}$ be the linear operator $\zeta \mapsto \big[t \mapsto A_{\xi,v}(t)\zeta(t)\big]$ with $A_{\xi,v}(t) = D_x v(t)\circ\xi(t)$. The function $A_{\xi,v}(t)$ is differentiable for any $t \in [-T;T]$ and $D_x A_{\xi,v}(t) = \big[D_x^2 v(t)\circ\xi(t)\big]\,[D_x\xi(t)]$. Therefore $A_{\xi,v}$ belongs to $\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and depends continuously on (ξ,v) (lemma 51). Consequently, $\mathcal{A}_{\xi,v}$ is a well-defined and continuous mapping from $\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ to $\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and the mapping $(\xi,v)\mapsto \mathcal{A}_{\xi,v}$ is continuous.

Let us prove the existence of $\partial_\xi H$ and the equality $\partial_\xi H(\xi,v)=\mathcal{A}_{\xi,v}$. We set

$$\Sigma = v\circ(\xi+\zeta) - v\circ\xi - D_x v\circ\xi\cdot\zeta \tag{A.11}$$

Straightforward calculations show that for any $i\in\{1,...,n\}$, $D_x\Sigma_i = D_x v_i\circ(\xi+\zeta)\cdot D_x(\xi+\zeta) - D_x v_i\circ\xi\cdot D_x\xi - \zeta^*\cdot[D_x^2 v_i\circ\xi]\cdot D_x\xi - D_x v_i\circ\xi\cdot D_x\zeta$ and therefore,

$$D_x\Sigma_i = K_i^*\cdot D_x\xi + L_i^*\cdot D_x\zeta \tag{A.12}$$

$$\text{with}\ \left\{\begin{array}{l} K_i = L_i - D_x^2 v_i\circ\xi\cdot\zeta \\ L_i = \nabla_x v_i\circ(\xi+\zeta) - \nabla_x v_i\circ\xi \end{array}\right.$$

We use the following intermediate result: we set $\varphi=v$ or $\varphi=\nabla_x v_i$ for a $i\in\{1,...,n\}$. Then, there is a function $\varepsilon:\mathbb{R}_+\to\mathbb{R}_+$, independent of (t,x), with $\lim_{h\to 0}\varepsilon(h)=0$ and such that for any $t\in[-T;T]$ and $x\in\mathbb{R}^n$, the inequality

$$\|\varphi(t)(x)-\varphi(t)(x+h)-D_x\varphi(t)(x)\cdot h\| = \varepsilon(h)\cdot\|h\| \tag{A.13}$$

holds. To prove this, we define $\psi(t,x,h)=\varphi(t)(x+h)-D_x\varphi(t)(x)\cdot(x+h)$ and notice that $D_h\psi(t,x,h) = D_x\varphi(t)(x+h)-D_x\varphi(t)(x)$. From the uniform equicontinuity of $(D_x\varphi(t))_{t\in[-T;T]}$ (see again the point *ii)* in the proof of the lemma 51), we deduce that there is a function $\lambda:\mathbb{R}_+\to\mathbb{R}_+$ such that $\lim_{h\to 0}\lambda(h)=0$ and $\|D_h\psi(t,x,h)\|\leq\lambda(h)$. Therefore

$$\begin{aligned}\varphi(t)(x)-\varphi(t)(x+h)-D_x\varphi(t)(x)\cdot h &= \psi(t,x,0)-\psi(t,x,h)\\ &= \int_0^1 [D_h\psi(t,x,\theta h)]\cdot h\,d\theta\end{aligned}$$

and consequently, the equation (A.13) holds with $\varepsilon(h)=\sup_{\delta\in[0,h]}\lambda(\delta)$.

From this result, we deduce the inequalities $\|\Sigma\|_{0,0}\leq\varepsilon(\|\zeta\|_{0,1})\cdot\|\zeta\|_{0,1}$, $\|K_i\|_{0,0}\leq\varepsilon(\|\zeta\|_{0,1})\cdot\|\zeta\|_{0,1}$ and $\|L_i\|_{0,0}\leq(\varepsilon(\|\zeta\|_{0,1})+\|v\|_{\mathcal{V}_2})\cdot\|\zeta\|_{0,1}$. Finally, using the expressions (A.11) and (A.12), we end up with

$$\|\Sigma\|_{0,0}+\|D_x\Sigma\|_{0,0}\leq\eta(\|\zeta\|_{0,1})\cdot\|\zeta\|_{0,1}\ \text{ with }\ \lim_{h\to 0}\eta(h)=0 \tag{A.14}$$

which proves the desired differentiability result.

• Existence and continuity of $\partial_v H$: the existence and expression of $\partial_v H(\xi,v)$ is clear: the mapping $[v\mapsto H(\xi,v)]$ is linear continuous and therefore $\partial_v H(\xi,v)\cdot W = H(\xi,W)$. The continuity of this partial derivative is a direct consequence of the lemma 51. ■

A.3.2 Differentiability of the flow

Let V be a vector field of $\mathcal{V}_1$. The corresponding flow is hereafter denoted ϕ, its dependance in V being implicit. It is the solution of the initial-value problem

$$\partial_t\phi = V\circ\phi\ \text{ and }\ \phi = I$$

We know that ϕ belongs to $\mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ (see [47]). The following proposition characterizes the regularity of the correspondence $V\to\phi$ under a stronger assumption on the regularity of V.

Proposition 53 *The mapping*

$$\begin{array}{ccc} \mathcal{V}_2 & \to & \mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n)) \\ V & \mapsto & \phi \end{array}$$

is continuously differentiable. For any $\delta V \in \mathcal{V}_2$, $\delta\phi = \partial_V \phi \cdot \delta V$ is the solution of

$$\partial_t(\delta\phi) = [D_x V \circ \phi] \cdot \delta\phi + \delta V \circ \phi \;\; \textit{and} \;\; \delta\phi(0) = 0 \tag{A.15}$$

Proof – For any $\xi \in \mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and $V \in \mathcal{V}_2$, we set

$$F(\xi, V) = \xi - \left[t \mapsto I + \int_0^t V(\tau) \circ \xi(\tau)\, d\tau\right] \tag{A.16}$$

The unique solution of $F(\xi, V) = 0$ is $\xi = \phi$. Moreover, from the lemma 52 we deduce that $(\xi, V) \mapsto F(\xi, V)$ is a $\mathcal{C}^1$ mapping from $\mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n)) \times \mathcal{V}_2$ to $\mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$. Its differential is given by:

$$\mathrm{d}F(\xi, v) \cdot (\zeta, \delta V) = \zeta - \left[t \mapsto \int_0^t \left([D_x V(\tau) \circ \xi(\tau)] \cdot \zeta(\tau) + \delta V(\tau) \circ \xi(\tau)\right)\, d\tau\right]$$

Therefore, $\partial_\xi F(\xi, V)$ is an isomorphism: for any $f \in \mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$, we set $\psi = \partial_t f$. The mapping f satisfies $\partial_\xi F(\xi, v) \cdot \zeta = f$ iff it is the solution of

$$\partial_t \zeta = [D_x V \circ \xi] \cdot \zeta + \psi \;\; \text{and} \;\; \zeta(0) = f(0) \tag{A.17}$$

Let

$$U(s,t) = \exp\left(\int_s^t D_x V(\tau) \circ \xi(\tau)\, d\tau\right) \tag{A.18}$$

The solution of the system (A.17) is given by

$$\zeta(t) = U(0,t) f(0) + \int_0^t U(t,\tau)\psi(\tau)\, d\tau$$

At that point, the regularity of $V \mapsto \phi$ appears as a simple consequence of the implicit function theorem. The expression $\delta\phi = \partial_V \phi \cdot \delta V$ is solution of $\partial_\xi F(\phi, V) \cdot (\delta\phi) = -\partial_V F(\phi, V) \cdot \delta V$ or equivalently, of the system (A.15). ■

Appendix B

The Oriented Distance Function

We provide in this appendix a brief account of the theory of the *oriented distanece function* and complement it by some specific results needed in the thesis. A complete review of the oriented distance function may be found in [19].

Distance Functions. Let A be a subset of $\mathbb{R}^n$. The *distance function* to A is defined by

$$d_A(x) = \begin{cases} \inf_{y \in A} \|x - y\|_{\mathbb{R}^n} & \text{if } A \neq \varnothing \\ +\infty & \text{if } A = \varnothing \end{cases} \tag{B.1}$$

where $\|\cdot\|_{\mathbb{R}^n}$ is the usual euclidean norm of $\mathbb{R}^n$. The *oriented distance function* to A is given by

$$b_A(x) = \begin{cases} d_A(x) & \text{if } x \in \complement A \\ -d_{\complement A}(x) & \text{if } x \in A \end{cases} \tag{B.2}$$

Projections. Let $h > 0$. The set $U_h(A)$ is the tubular neighbourghood of thickness h of the set A:

$$U_h(A) = A + B(0, h) = \{x \in \mathbb{R}^n,\ d_A(x) < h\} \tag{B.3}$$

The multivalued mapping $\Pi : \mathbb{R}^n \rightsquigarrow \partial A$, defined by

$$\Pi(x) = \{p \in \partial A,\ \|p - x\|_{\mathbb{R}^n} = |b_A(x)|\} \tag{B.4}$$

is the *projection mapping* on ∂A. When the set $\Pi(x)$ is reduced to one element, we set

$$\{p(x)\} = \Pi(x) \tag{B.5}$$

Smoothness of sets. Let $C(x, r)$ be the hypercube of $\mathbb{R}^n$ centered at x with edge of length r. An open set A of $\mathbb{R}^n$ is *of class* $\mathcal{C}^{k,\lambda}$ if for any $x \in \partial A$, there is a $r > 0$ and a function $f \in \mathcal{C}^{k,\lambda}(\mathbb{R}^{n-1}; \mathbb{R})$ such that, after a suitable rotation of the axes, we have

$$A \cap C(x, r) = \{y \in \mathbb{R}^n,\ f(y_1, ..., y_{n-1}) < y_n\} \cap C(x, r) \tag{B.6}$$

Characterization of the smoothness by the distance function.

Theorem 7 *Let A be a bounded open subset of $\mathbb{R}^n$. The set A is of class $\mathcal{C}^{k,\lambda}$ with $k = 1$ and $\lambda = 1$ or $k \geq 2$ and $\lambda \in [0,1]$ if and only if the restriction of b_A to $U_h(\partial A)$ is of class $\mathcal{C}^{k,\lambda}$.*

For any such h, the mapping $x \mapsto p(x) : U_h(\partial A) \to \partial A$ is of class $\mathcal{C}^{k-1,\lambda}$ and satisfies

$$p(x) = x - b_A(x) \cdot \nabla b_A(x) \tag{B.7}$$

Moreover, the unit outer normal n to A, defined on ∂A, satisfies

$$n \circ p = \nabla b_A \;\text{ on } U_h(\partial A) \tag{B.8}$$

Tangential Sobolev spaces. Let A be an open bounded set of $\mathbb{R}^n$ of class $\mathcal{C}^2$ and ∂A its boundary. Let $h > 0$ be as in the theorem 7.

Definition 25 *Let $f : \partial A \to \mathbb{R}$. By definition*

$$f \in W^{1,p}(\partial A) \;\Leftrightarrow\; \exists\, k \in (0,h],\; f \circ p \in W^{1,p}(U_k(\partial A)) \tag{B.9}$$

For any such f, there is a unique mapping $\nabla_\tau f \in L^p(\partial A)^n$ such that

$$\nabla(f \circ p) = [I - b_A \cdot D^2 b_A] \cdot \nabla_\tau f \circ p \tag{B.10}$$

This mapping is the tangential gradient *of f.*

At that point, it is straightforward to define the *tangential Jacobian matrix* $D_\tau f$ and the *tangential divergence* $\operatorname{div}_\tau f$ for any vectorial mapping $f \in W^{1,p}(\Gamma)^n$. We set

$$D_\tau f \circ p \cdot [I - b_A \cdot D^2 b_A] = D(f \circ p) \;\text{ and }\; \operatorname{div}_\tau f = \operatorname{tr} D_\tau f \tag{B.11}$$

These three definitions are consistent with the classical definitions used for mappings that are continuously differentiable on ∂A.

Federer formula.

Proposition 54 *Let Ω be an open bounded $\mathcal{C}^{1,1}$ set of $\mathbb{R}^n$. Let $h > 0$ and $\varepsilon > 0$ be such that $b_\Omega \in \mathcal{C}^{1,1}(U_{h+\varepsilon})$. The equations*

$$T_z(x) = x + z\nabla b(x) \;\text{ and }\; j_z(x) = \det(DT_z(x)) \tag{B.12}$$

define the mappings $T_z : \Gamma \to U_{h+\varepsilon}$ and $j_z : \Gamma \to \mathbb{R}$ for any $z \in [-h;h]$. Moreover

(i) *the mapping $T : (x,z) \in \Gamma \times [-h;h] \mapsto T_z(x)$ and its inverse are Lipschitz.*

(ii) *for any $z \in [-h;h]$, $j_z \in L^\infty(\Gamma; \mathcal{H}^{n-1})$.*

(iii) Federer: *for any $F \in L^1(U_h(\Gamma))$*

$$\int_{U_h(\Gamma)} F\,dx = \int_{-h}^{h}\int_\Gamma F \circ T_z \cdot j_z\, d\mathcal{H}^{n-1} dz = \int_\Gamma \int_{-h}^{h} F \circ T_z \cdot j_z\, dz d\mathcal{H}^{n-1} \tag{B.13}$$

Tangential Stokes formula.

Lemma 53 *Let Ω be a bounded open set of $\mathbb{R}^n$ of class $\mathcal{C}^2$. For any $f \in W^{1,1}(\Gamma;\mathbb{R})$ and $V \in \mathcal{C}^1(\Gamma;\mathbb{R}^n)$, we have*

$$\int_\Gamma f \cdot \mathrm{div}_\tau V \, d\mathcal{H}^{n-1} = -\int_\Gamma \langle \nabla_\tau f, V\rangle \, d\mathcal{H}^{n-1} + (n-1)\int_\Gamma Hf \, \langle V, n\rangle \, d\mathcal{H}^{n-1} \quad \text{(B.14)}$$

Proof – Let us assume first that the field V is tangent to Γ, that is $\langle V, n\rangle \equiv 0$ where n is the outer unitary normal to Ω. For the sake of shortness, we use b instead of b_Ω. Let $h > 0$ be as in the proposition 54. For any $0 < k \leq h$, we define the function V_k by

$$V_k = \xi_k \circ b \cdot V \circ p \quad \text{(B.15)}$$

where $\xi_k(x) = \xi(x/k)$, ξ being such that

$$\xi \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}), \ \mathrm{Supp}\, \xi \subset [-1,1] \ \text{ and } \ \int_{\mathbb{R}} \xi(x)\, dx = 1 \quad \text{(B.16)}$$

We also set $F = f \circ p$. As $f \in W^{1,1}(\Gamma;\mathbb{R})$, the function F belongs to $W^{1,1}(U_k;\mathbb{R})$ for a given $k \in (0,h]$ and we have

$$A_k := \int_{U_k(\Gamma)} F \cdot \mathrm{div}\, V_k \, dx = -\int_{U_k(\Gamma)} \langle \nabla F, V_k\rangle \, dx =: B_k \quad \text{(B.17)}$$

By the proposition 54, we obtain

$$\begin{aligned} A_k &= \int_{U_k(\Gamma)} f \circ p \cdot \mathrm{div}\, V_k \, dx \\ &= \int_{-k}^{k} \left[\int_\Gamma f \cdot (\mathrm{div}\, V_k) \circ T_z \cdot j_z \, d\mathcal{H}^{n-1} \right] dz \end{aligned}$$

and therefore

$$A_k = \int_\Gamma f \cdot C(k,\cdot)\, d\mathcal{H}^{n-1} \ \text{ with } \ C(k,x) = \int_{-k}^{k} (\mathrm{div}\, V_k) \circ T_z(x) \cdot j_z(x)\, dz \quad \text{(B.18)}$$

Let us evaluate $\mathrm{div}\, V_k$. We have clearly

$$DV_k = \xi_k \circ b \cdot D(V \circ p) + \xi_k' \circ b \cdot V \circ p \cdot Db \quad \text{(B.19)}$$

As $\nabla b = n$ on Γ, for any $x \in \Gamma$, $V \circ p(x) \cdot Db(x) = \langle V(x), n(x)\rangle$. As V is tangent, the second term of the right-hand side of (B.19) vanishes. On the other hand, $D(V \circ p) = (D_\tau V) \circ p \cdot [I - b \cdot D^2 b]$. Therefore, using the identity $I - z \cdot D^2 b \circ T_z = [I + z \cdot D^2 b]^{-1}$ ([19], th. 7.1, p. 42), we end up with

$$(\mathrm{div}\, V_h) \circ T_z \cdot j_z = \xi_h(z) \cdot \mathrm{tr}((D_\tau V) \cdot [I + zD^2 b]^{-1}) \cdot \det(I + zD^2 b)$$

(we also used the identity $j_z = \det(I + z \cdot D^2 b)$, see [19],th. 7.1 p. 42). $D^2 b$ belongs to $L^\infty(\Gamma; \mathcal{H}^{n-1})$ (proposition 54) and consequently $z \cdot D^2 b \to 0$ in $L^\infty(\Gamma; \mathcal{H}^{n-1})$ when $z \to 0$. Therefore, the mapping

$$\varepsilon_z := \mathrm{tr}((D_\tau V) \cdot [I + zD^2 b]^{-1}) \cdot \det(I + zD^2 b) - \mathrm{div}_\tau V \quad \text{(B.20)}$$

satisfies

$$\sup_{z\in[-k;k]} \|\varepsilon_z\|_{L^\infty(\Gamma)} \to 0 \text{ when } k \to 0$$

And, as a consequence of (B.16)

$$\|C(k,\cdot) - k\cdot \mathrm{div}_\tau V\|_{L^\infty(\Gamma)} = \left\|\int_{-k}^{k} \xi_h(z)\cdot \varepsilon_z(x)\,dz\right\|_{L^\infty(\Gamma)} = \mathsf{o}(k) \text{ when } k \to 0$$

This result, used in (B.18), yields

$$A_k = k\cdot \int_\Gamma f\cdot \mathrm{div}_\tau V\, d\mathcal{H}^{n-1} + \mathsf{o}(k) \text{ when } k\to 0 \tag{B.21}$$

We perform a similar computation for B_k. As $\nabla F = [I - b\cdot D^2 b](\nabla_\tau f)\circ p$, using again the the formula (B.13) of the proposition 54, we obtain

$$\begin{aligned} B_k &= -\int_{U_k} \xi_k\circ b\cdot \left\langle [I - b\cdot D^2 b](\nabla_\tau f)\circ p, V\circ p\right\rangle dx \\ &= -\int_{-k}^{k}\left[\int_\Gamma \xi_k(z)\left\langle [I + zD^2 b]^{-1}\nabla_\tau f, V\right\rangle j_z\, d\mathcal{H}^{n-1}\right] dz \\ &= -\int_\Gamma \langle \nabla_\tau f, D(k,\cdot)V(x)\rangle\, d\mathcal{H}^{n-1} \end{aligned}$$

with

$$D(k,x) = \int_{-k}^{k} \xi_h(z)[I + z\cdot D^2 b(x)]^{-1} \det(I + z\cdot D^2 b(x))\,dz$$

We find easily that

$$\|D(k,\cdot) - kI\|_{L^\infty(\Gamma)} = \mathsf{o}(k) \text{ when } k\to 0$$

and therefore that

$$B_k = -k\cdot \int_\Gamma \langle \nabla_\tau f, V\rangle\, d\mathcal{H}^{n-1} + \mathsf{o}(k) \tag{B.22}$$

The equality $A_k = B_k$ and the equations (B.21) and (B.22) provide the equality (B.14) when V is tangent to Γ.

To prove this equality in the general case, *i.e.* when V is not tangent to Γ, we use the previous formula in the tangent case with

$$V_\tau = V - \langle V, n\rangle\, n \tag{B.23}$$

On one hand, we have

$$\begin{aligned} \mathrm{div}_\tau V_\tau &= \mathrm{div}_\tau V - \mathrm{div}_\tau(\langle V,n\rangle\, n) \\ &= \mathrm{div}_\tau V - \langle \nabla_\tau \langle V,n\rangle, n\rangle - \langle V,n\rangle\, \mathrm{div}_\tau n \\ &= \mathrm{div}_\tau V - \langle V,n\rangle\, \Delta b \\ &= \mathrm{div}_\tau V - (n-1)H\, \langle V,n\rangle \end{aligned}$$

and on the other hand,

$$\langle \nabla_\tau f, V_\tau\rangle = \langle \nabla_\tau f, V\rangle \quad \text{as} \quad \langle \nabla_\tau f, n\rangle \equiv 0$$

Finally, the equation

$$\int_\Gamma f \cdot \mathrm{div}_\tau V_\tau \, d\mathcal{H}^{n-1} = -\int_\Gamma \langle \nabla_\tau f, V_\tau \rangle \, d\mathcal{H}^{n-1}$$

yields (B.14). ■

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