Coriolis and Centrifugal Forces

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Euler-Lagrange Dynamics

The dynamics of a mechanical system with coordinates \( q \), kinetic energy \( K \) and potential energy \( V \) which is subject to the forces \( f \) is governed by the Euler-Lagrange equations

\[
\frac{d}{dt} \nabla_q L(q, \dot{q}) - \nabla_q L(q, \dot{q}) = f
\]

where the Lagrangian \( L \) of the system is defined as

\[
L(q, \dot{q}) = K(q, \dot{q}) - V(q).
\]

Since the kinematic energy varies quadratically with the velocity,

\[
K(q, \dot{q}) = \frac{1}{2} q^t M(q) \dot{q}
\]

for some mass matrix \( M(q) \) which is symmetric and positive (semi-)definite. Using this expression in the Euler-Lagrange equations provides

\[
M(q) \ddot{q} = f - \nabla_q V(q) + f^c
\]

where \( f^c \) are – by definition – the Coriolis and centrifugal forces:

\[
f^c = -\dot{M}(q, \dot{q}) \dot{q} + \frac{1}{2} \nabla_q (q^t M(q) \dot{q})
\]
Coriolis and Centrifugal Forces

Since \( f^c = -\dot{M}(q, \dot{q})\dot{q} + \frac{1}{2} \nabla_q (q^t M(q)\dot{q}) \), every component \( f^c_i \) of the Coriolis and centrifugal forces varies quadratically with the velocity: we can find some matrix \( \Gamma_i \) such that

\[
 f^c_i = -\dot{q}^t \Gamma_i \dot{q} = -\sum_{j,k} \dot{q}_j \Gamma_{ijk} \dot{q}_k
\]

Indeed, we have

\[
 f^c_i = -\sum_j \dot{M}_{ij}\dot{q}_j + \frac{1}{2} \frac{\partial}{\partial q_i} \sum_{j,k} \dot{q}_j M_{jk} \dot{q}_k
\]

and thus

\[
 f^c_i = \sum_{j,k} \left( -\frac{\partial M_{ij}}{\partial q_k} \dot{q}_k \dot{q}_j + \frac{1}{2} \frac{\partial M_{jk}}{\partial q_i} \dot{q}_k \dot{q}_j \right)
\]

Swapping the indices \( j \) and \( k \) yields

\[
 \sum_j \sum_k \frac{\partial M_{ij}}{\partial q_k} \dot{q}_k \dot{q}_j = \sum_k \sum_j \frac{\partial M_{ik}}{\partial q_j} \dot{q}_j \dot{q}_k = \sum_j \sum_k \frac{\partial M_{ik}}{\partial q_j} \dot{q}_k \dot{q}_j
\]

and thus we may also write

\[
 f^c_i = -\sum_{j,k} \frac{1}{2} \left( \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i} \right) \dot{q}_k \dot{q}_j
\]

hence we may define

\[
 \Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i} \right)
\]

These \( \Gamma_{ijk} \) are called the Christoffel symbols of the first kind of \( M(q) \).

Mechanical Energy

Let \( C(q, \dot{q}) \) be the matrix defined as

\[
 C_{ij} = \sum_k \Gamma_{ijk} \dot{q}_k.
\]

By construction \( f_c = -C(q, \dot{q})\dot{q} \) thus the Euler-Lagrange equations become

\[
 M(q)\ddot{q} + C(q, \dot{q})\dot{q} = f - \nabla_q V(q).
\]
Consider the matrix $S := \dot{M} - 2C$. Since

$$S_{ij} = [\dot{M} - 2C]_{ij} = \sum_k \frac{\partial M_{ik}}{\partial q_k} \dot{q}_k - \sum_k \left( \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i} \right) \dot{q}_k$$

$$= \sum_k \left( \frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{ik}}{\partial q_j} \right) \dot{q}_k$$

it is skew-symmetric:

$$\forall i, \forall j, S_{ij} = -S_{ji}$$

or equivalently$^1$,

$$\forall v \in \mathbb{R}^n, v^t S v = 0.$$ 

The mechanical energy of a Euler-Lagrange systems is defined as

$$E(q, \dot{q}) = K(q, \dot{q}) + V(q).$$

It time derivative satisfies

$$\dot{E} = \dot{q}^t M(q) \ddot{q} + \frac{1}{2} \dot{q}^t \dot{M}(q) \ddot{q} + \nabla V(q) \dot{q}$$

and thus

$$\dot{E} = \dot{q}^t (-C(q, \dot{q}) \ddot{q} + f - \nabla_q V(q)) + \frac{1}{2} \dot{q}^t \dot{M}(q) \ddot{q} + \nabla V(q) \dot{q}$$

$$= f^t \dot{q} + \frac{1}{2} \ddot{q} S \dot{q}$$

and since $S$ is skew-symmetric

$$\dot{E} = f \cdot \dot{q}.$$ 

In plain words: the derivative of the mechanical energy is equal to the power transferred to the system by non-conservative external forces.

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$^1$If $v^t S v = 0$ for every $v$, then $v = e_i$ yields $S_{ii} = 0$ and $v = e_i + e_j$ yields $S_{ij} + S_{ji} = 0$ and $S_{ji} = -S_{ij}$. On the other hand, if $S_{ij} = -S_{ji}$ for every $i$ and $j$, then since every vector $v$ may be represented as a combination $v = \sum_i \lambda_i e_i$, we have

$$v^t S v = \sum_{i,j} S_{ij} \lambda_i \lambda_j = \sum_{i < j} S_{ij} \lambda_i \lambda_j + \sum_i S_{ii} (\lambda_i)^2 + \sum_{i > j} S_{ij} \lambda_i \lambda_j$$

$$= \sum_{i < j} S_{ij} \lambda_i \lambda_j + \sum_i 0 \times (\lambda_i)^2 - \sum_{j < i} S_{ji} \lambda_j \lambda_i$$

$$= 0.$$