Coriolis and Centrifugal Forces

Sébastien Boisgérault, MINES ParisTech, under CC BY-NC-SA 4.0

November 8, 2017

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Euler-Langrange Dynamics

The dynamics of a mechanical system with coordinates q, kinetic energy K and potential energy V which is subject to the forces f is governed by the Euler-Lagrange equations

$$\frac{d}{dt}\nabla_{\dot{q}} L(q, \dot{q}) - \nabla_q L(q, \dot{q}) = f$$

where the Lagrangian L of the system is defined as

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q).$$

Since the kinematic energy varies quadratically with the velocity,

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^t M(q)\dot{q}$$

for some mass matrix $M(q)$ which is symmetric and positive (semi-)definite. Using this expression in the Euler-Lagrange equations provides

$$M(q)\ddot{q} = f - \nabla_q V(q) + f^c$$

where f^c are – by definition – the Coriolis and centrifugal forces:

$$f^c = -\dot{M}(q, \dot{q})\dot{q} + \frac{1}{2}\nabla_q(\dot{q}^t M(q)\dot{q})$$

Coriolis and Centrifugal Forces

Since $f^c = -\dot{M}(q,\dot{q})\dot{q} + \frac{1}{2}\nabla_q(\dot{q}^t M(q)\dot{q})$, every component f^c_i of the Coriolis and centrifugal forces varies quadratically with the velocity: we can find some matrix Γ_i such that

$$f^c_i = -\dot{q}^t \Gamma_i \dot{q} = -\sum_{j,k} \dot{q}_j \Gamma_{ijk} \dot{q}_k$$

Indeed, we have

$$f^c_i = -\sum_j \dot{M}_{ij}\dot{q}_j + \frac{1}{2}\frac{\partial}{\partial q_i}\sum_{j,k}\dot{q}_j M_{jk}\dot{q}_k$$

and thus

$$f^c_i = \sum_{j,k}\left(-\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j + \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_k\dot{q}_j\right)$$

Swapping the indices j and k yields

$$\sum_j\sum_k \frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j = \sum_k\sum_j \frac{\partial M_{ik}}{\partial q_j}\dot{q}_j\dot{q}_k = \sum_j\sum_k \frac{\partial M_{ik}}{\partial q_j}\dot{q}_k\dot{q}_j$$

and thus we may also write

$$f^c_i = -\sum_{j,k}\frac{1}{2}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_k\dot{q}_j$$

hence we may define

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right)$$

These Γ_{ijk} are called the Christoffel symbols of the first kind of $M(q)$.

Mechanical Energy

Let $C(q,\dot{q})$ be the matrix defined as

$$C_{ij} = \sum_k \Gamma_{ijk}\dot{q}_k.$$

By construction $f_c = -C(q,\dot{q})\dot{q}$ thus the Euler-Lagrange equations become

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = f - \nabla_q V(q).$$

Consider the matrix $S := \dot{M} - 2C$. Since

$$S_{ij} = [\dot{M} - 2C]_{ij} = \sum_k \frac{\partial M_{ij}}{\partial q_k} \dot{q}_k - \sum_k \left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i} \right) \dot{q}_k$$
$$= \sum_k \left(\frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{ik}}{\partial q_j} \right) \dot{q}_k$$

it is skew-symmetric:

$$\forall\, i,\ \forall\, j,\ S_{ij} = -S_{ji}$$

or equivalently[1],

$$\forall\, v \in \mathbb{R}^n,\ v^t S v = 0.$$

The mechanical energy of a Euler-Lagrange systems is defined as

$$E(q, \dot{q}) = K(q, \dot{q}) + V(q).$$

It time derivative satisfies

$$\dot{E} = \dot{q}^t M(q) \ddot{q} + \frac{1}{2} \dot{q}^t \dot{M}(q) \dot{q} + \nabla V(q) \dot{q}$$

and thus

$$\dot{E} = \dot{q}^t \left(-C(q, \dot{q})\dot{q} + f - \nabla_q V(q) \right) + \frac{1}{2} \dot{q}^t \dot{M}(q) \dot{q} + \nabla V(q) \dot{q}$$
$$= f^t \dot{q} + \frac{1}{2} \dot{q}^t S \dot{q}$$

and since S is skew-symmetric

$$\dot{E} = f \cdot q.$$

In plain words: the derivative of the mechanical energy is equal to the power transferred to the system by non-conservative external forces.

[1] If $v^t S v = 0$ for every v, then $v = e_i$ yields $S_{ii} = 0$ and $v = e_i + e_j$ yields $S_{ii} + S_{jj} + S_{ij} + S_{ji} = 0$ and thus $S_{ij} = -S_{ji}$. On the other hand, if $S_{ij} = -S_{ji}$ for every i and j, then since every vectory v may be represented as a combination $v = \sum_i \lambda_i e_i$, we have

$$v^t S v = \sum_{i,j} S_{ij} \lambda_i \lambda_j = \sum_{i<j} S_{ij} \lambda_i \lambda_j + \sum_i S_{ii} (\lambda_i)^2 + \sum_{i>j} S_{ij} \lambda_i \lambda_j$$
$$= \sum_{i<j} S_{ij} \lambda_i \lambda_j + \sum_i 0 \times (\lambda_i)^2 - \sum_{j<i} S_{ji} \lambda_j \lambda_i$$
$$= 0.$$