Introduction to Delay Systems

Sébastien Boisgérault
<Sebastien.Boisgerault@mines-paristech.fr>

Centre de Robotique / CAOR,
Math. et Systèmes, Mines ParisTech

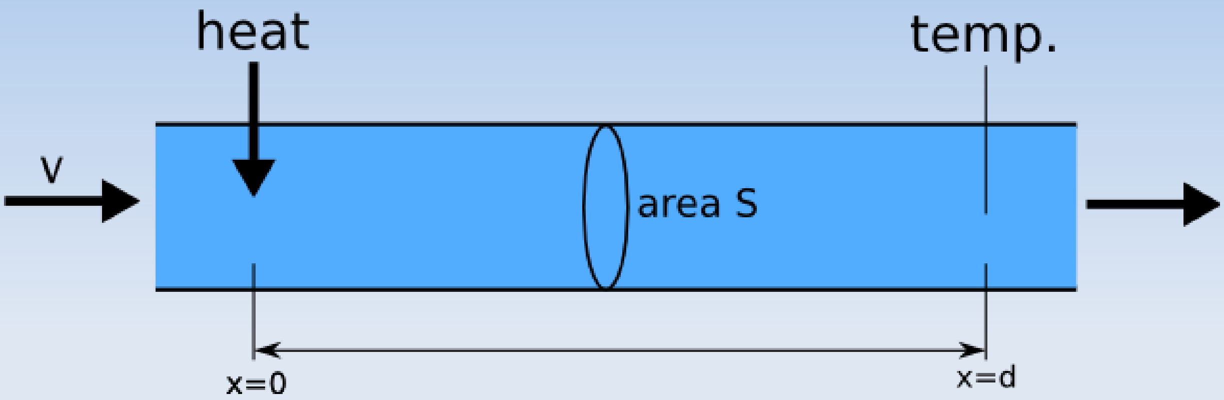
Prerequisites

- Control Theory:
  - Finite-Dim. **L**inear **T**ime-**I**nvariant Systems,
  - State-Space and Input/Output settings.
- Mathematics:
  - Measure Theory,
  - Linear Operators,
  - Complex Analysis,
  - ~~Semigroup Theory~~.

Introduction

- Sometimes ordinary differential equations (**ODE**s) are not the proper model for a system:
  - delays are necessary $\rightarrow$ **FDE**s (Functional)
- hopefully such models are:
  - linear and time-invariant (LTI),
- but their state space is infinite-dimensional :
  - new modelling, analysis and control methods are required.

Example: Control of a Thermic Loop



System properties:

- volumic heat capacity: c,
- thermal conductivity: κ.

Thermic Loop: Analytic Model I

Let the reference temperature be 0,

Inject Q joules at $t = 0,\ x = 0$

$$T(t = 0^+, x) = \frac{Q}{cS}\delta_0(x)$$

$$\left(\int_{-\infty}^{+\infty} c \cdot T(t = 0^+, x)\, S dx = Q\right)$$

Thermic Loop: Analytic Model II

Heat Diffusion

$$\left|\begin{array}{r} j = -\kappa \nabla T \\ c\partial T/\partial t = -\nabla \cdot j \end{array}\right. \qquad \frac{\partial T}{\partial t} - \alpha \Delta T = 0, \;\; \alpha = \frac{\kappa}{c}$$

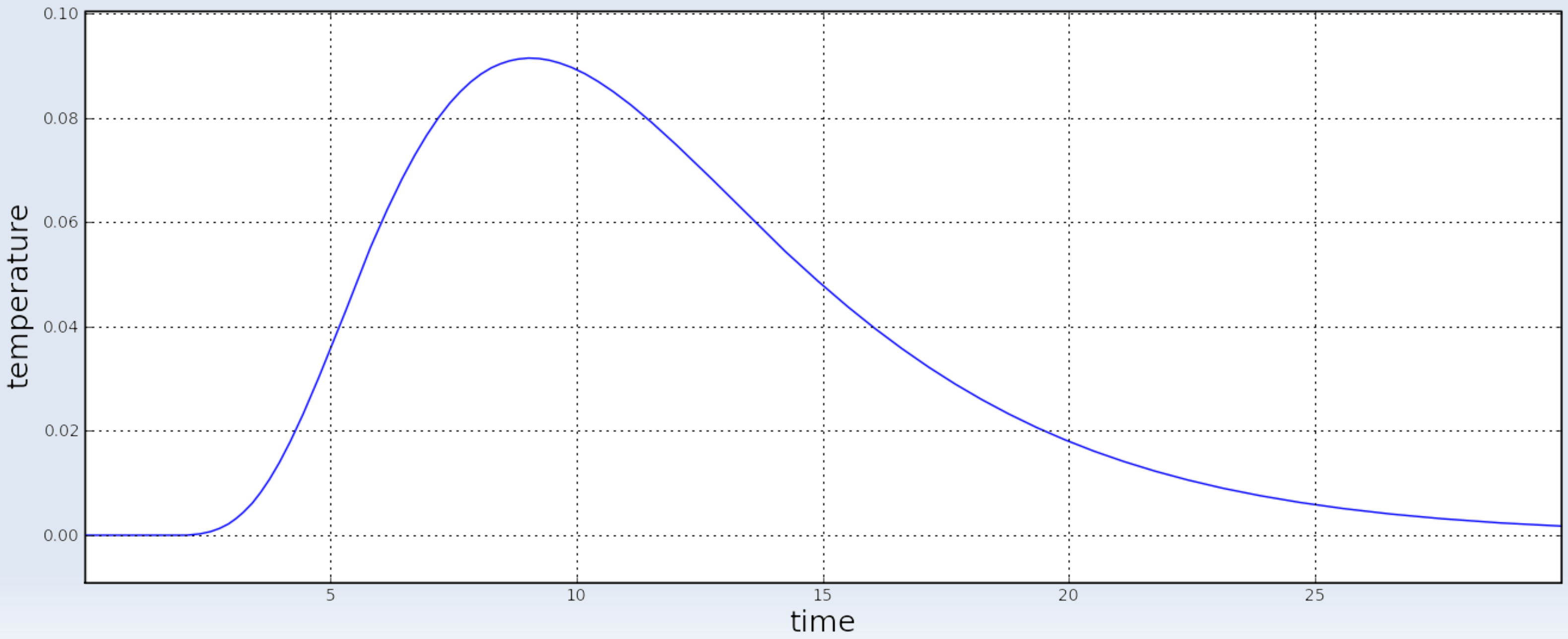
Impulse response (1 joule injected)

$$T(t,x) = \frac{1}{cS\sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right)$$

Thermic Model: Analytic Model III

Take into account the transport

$$T(t,d) = \frac{1}{cS\sqrt{4\pi\alpha t}} \exp\left(-\frac{(d-vt)^2}{4\alpha t}\right)$$



Identification: System Structure

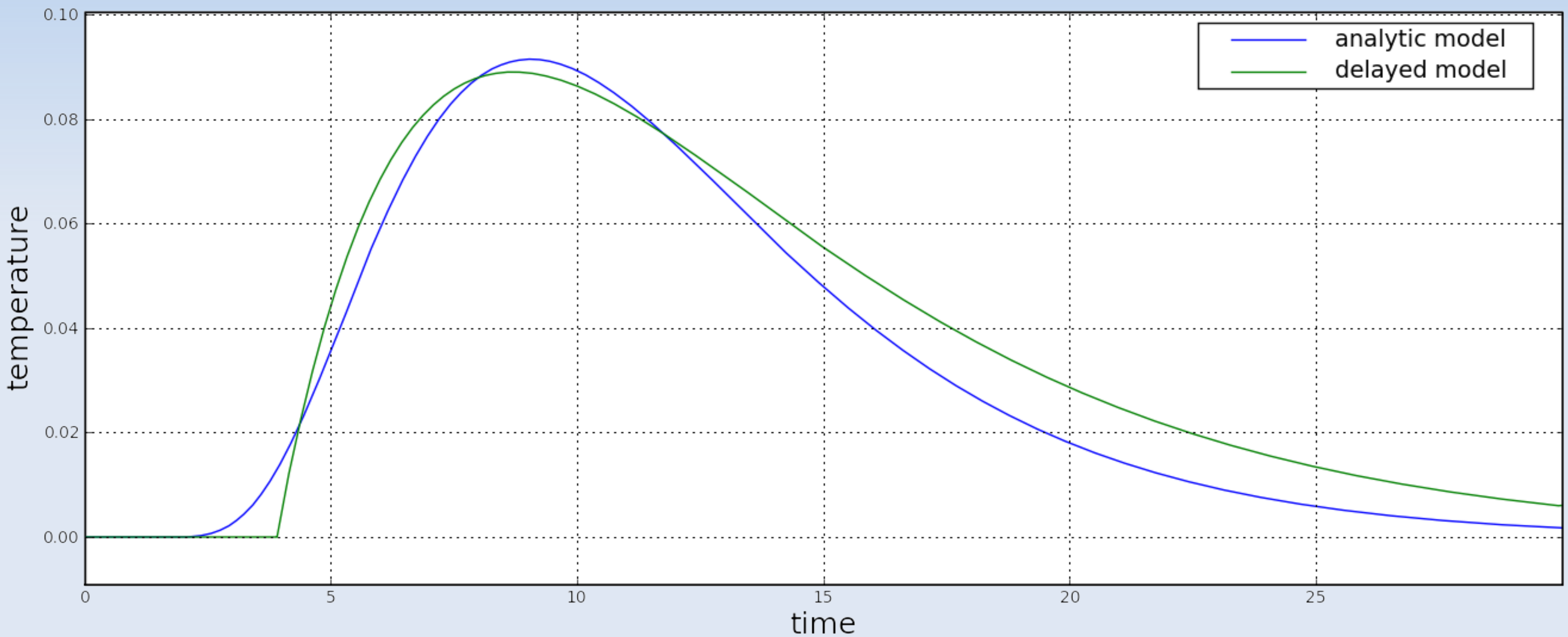
Delay: T

Two time scales: $\tau_1 > \tau_2$

$$h(t) = \left| \begin{array}{ll} 0 & \text{for } t < T \\ A\left(\exp\left(-\frac{t-T}{\tau_1}\right) - \exp\left(-\frac{t-T}{\tau_2}\right)\right) & \text{otherwise} \end{array} \right.$$

$$H(s) = \frac{a}{s^2 + bs + c}\exp(-sT)$$

Identification:
2nd-order + delay



3rd-order linear model improves accuracy

Identification: State-Space Model

- State-space model for this **dead-time system**:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bq(t) \\ T(t) = Cx(t-T) \end{cases} \qquad x(t) \in \mathbb{R}^2$$

- Control: try a linear feedback: $q(t) = -KT(t)$

$$\rightarrow \quad \dot{x}(t) = Ax(t) - BKCx(t-T)$$

What are the properties of this system ?

Delay-Differential Systems: Examples

$$\dot{x}(t) = \frac{1}{2}x(t) + \frac{1}{2}x(t-T)$$

$$\dot{x}(t) = \frac{1}{T}\int_{[-T,0]} x(t+\theta)\,d\theta$$

$$\dot{x}(t) = \frac{1}{T}(x(t) - 2x(t-T/2) + x(t-T))$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t-T) \\ x_1(t-T) \end{bmatrix}$$

State-Space Model: Concrete Framework

Concrete Framework (Retarded Equation)

$\mathcal{M}$: set of signed Borel measures on $[-T, 0]$

$x \in \mathbb{R}^n, \ A \in \mathcal{M}^{n \times n}$

$$\dot{x}(t) = \int_{[-T,0]} dA(\theta) x(t+\theta)$$

$$\downarrow$$

$$\dot{x}_i(t) = \sum_{j=1}^{n} \int_{[-T,0]} x_j(t+\theta)\, dA_{ij}(\theta), \ i \in \{1, \cdots, n\}$$

System State Space

State for $t = 0$: $x|_{[-T;0]} : [-T, 0] \to \mathbb{R}^n$

For $t \geq 0$ define $x_t : [-T, 0] \to \mathbb{R}^n$ by:

$$x_t(\theta) = x(t + \theta)$$

State Space X of continuous functions:

$$X = C^0([-T, 0]; \mathbb{R}^n)$$

$$||x||_X = \sup_{\theta \in [-T, 0]} ||x(\theta)||_{\mathbb{R}^n}$$

State-Space Model: Abstract Framework

$$\left| \begin{array}{rcl} \dot{x}(t) & = & Ax_t,\ t > 0 \\ x_0 & = & \phi \in X \end{array} \right.$$

with $A \in \mathcal{L}(X, \mathbb{R}^n)$: delay operator.

$\mathcal{L}(X, \mathbb{R}^n)$: linear continous operators $X \to \mathbb{R}^n$

State-Space Model Unified Framework

$\mathcal{M}$: set of signed Borel measures on $[-T, 0]$

Riesz Representation Theorem:

$$A \in \mathcal{L}(X, \mathbb{R}^n)$$

if and only iff there is $A \in \mathcal{M}^{n \times n}$ such that

$$A\phi = \int_{[-T,0]} dA(\theta)\phi(\theta)$$

Ordinary Differential Equations: Initial Value (Cauchy) Problem

$$\left| \begin{array}{l} \dot{x}(t) = f(t, x(t)),\ t > 0, \qquad f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \\ x(0) = \phi \in \mathbb{R}^n \end{array} \right.$$

$$x(t) = \phi + \int_0^t f(\tau, x(\tau))\, d\tau$$

Caratheodory - Existence + Uniqueness:

$$\forall\, x \in \mathbb{R}^n,\ t \mapsto f(t, x) \text{ measurable}$$

$$\exists\, \alpha \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}),\ \|f(t, 0)\|_{\mathbb{R}^n} \leq \alpha(t)$$

$$\exists\, \beta \in L^1_{\text{loc}}(\cdots),\ \|f(t, x) - f(t, y)\|_{\mathbb{R}^n} \leq \beta(t)\|x - y\|_{\mathbb{R}^n}$$

Differential-Delay Systems Initial Value (Cauchy) Problem

$$f : \mathbb{R}_+ \times X \to \mathbb{R}^n$$

$$\left| \begin{array}{rcl} \dot{x}(t) & = & f(t, x_t),\ t > 0, \\ x_0 & = & \phi \in X \end{array} \right.$$

Caratheodory - Existence + Uniqueness:

$$\forall\, x \in X,\ t \mapsto f(t, x) \text{ measurable}$$

$$\exists\, \alpha \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}),\ \|f(t, 0)\|_{\mathbb{R}^n} \leq \alpha(t)$$

$$\exists\, \beta \in L^1_{\text{loc}}(\cdots),\ \|f(t, x) - f(t, y)\|_{\mathbb{R}^n} \leq \beta(t)\|x - y\|_X$$

Linear DDS

$$\left| \begin{array}{l} \dot{x}(t) = Ax_t, \; t > 0, \\ x_0 = \phi \in X \end{array} \right.$$

$$A \in \mathcal{L}(X, \mathbb{R}^n)$$

- Solution existence and uniqueness:
  - in general, check Caratheodory solutions (fixed-point argument behind the scenes),
  - in simple cases « method of steps » works.

Linear Delay-Differential Algebraic System

$$\left|\begin{aligned} \dot{x}(t) &= Ax_t + By_t \\ y(t) &= Cx_t + Dy_t \end{aligned}\right.$$

delay operators: $A, B, C, D \in \mathcal{L}(C^0([-T, 0]; \mathbb{R}^{\cdots}); \mathbb{R}^{\cdots})$

$$x_0 \in C^0([-T, 0]; \mathbb{R}^n), y_0 \in C^0([-T, 0]; \mathbb{R}^m)$$

Existence and uniqueness if:

no algebraic loop: $I - D(\{0\})$ invertible

consistency: $y_0(0) = Cx_0 + Dy_0$

Delay Systems – Stability

Internal/External Stability

Internal Stability

$$\left|\begin{array}{l}\dot{x}(t) = Ax_t,\ t > 0 \\ x_0 = \phi\end{array}\right.$$

- **stable**:

$$\forall \varepsilon > 0,\ \exists \delta > 0, \|\phi\|_X < \delta \Rightarrow \sup_{t>0} \|x(t)\|_{\mathbb{R}^n} < \varepsilon$$

$$\text{or } \forall \varepsilon > 0,\ \exists \delta > 0, \|\phi\|_X < \delta \Rightarrow \sup_{t>0} \|x_t\|_X < \varepsilon$$

- **asympt. stable**: stable and

$$x(t) \text{ (or } \|x_t\|_X) \to 0 \text{ when } t \to +\infty$$

- **exp. stable**: $\exists\, K > 0, \alpha > 0$ such that

$$|x(t)| \text{ (or } \|x_t\|_X) \leq K\|\phi\|_X \exp(-\alpha t)$$

External (I/O) Stability

L^2 stability:

$$\left| \begin{array}{rcl} \dot{x}(t) & = & Ax_t + Bu_t \\ y(t) & = & Cx_t + Du_t \end{array} \right. \qquad x_0 = 0$$

$$\exists\, K,\ \forall u \in L^2(\mathbb{R}_+),\ \int |y(t)|^2\, dt \leq K^2 \int |u(t)|^2\, dt$$

Delays as a Convolutions

- Consider the delay operator $u \to y$ with:

$$y(t) = Au_t = \int_{[-T,0]} dA(\theta)\, u(t+\theta)$$

- Let B be a compact Borel subset of $\mathbb{R}$.

$$A^*(B) = A(-B) = \int_{[-T,0]} \chi_B(-\theta)\, dA(\theta)$$

defines a matrix-valued measure A^* on $[0, T]$.

Examples

- Pointwise delay: $y(t) = u(t - T)$

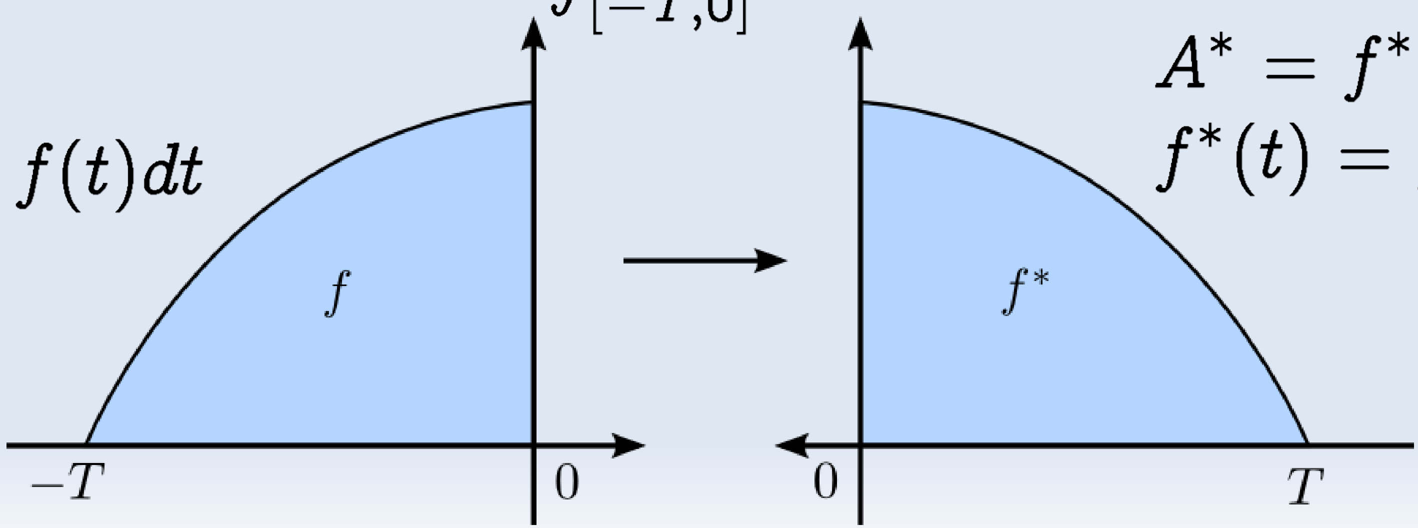
$$A = \delta(t + T) \;\rightarrow\; A^* = \delta(t - T)$$

- Distributed delay: $f \in L^1(-T, 0; \mathbb{R})$

$$y(t) = \int_{[-T,0]} [f(\theta)d\theta]u_t(\theta)$$

$$A = f(t)dt$$

$$A^* = f^*(t)dt$$
$$f^*(t) = f(-t)$$



- Every delay operator is a convolution:

$$y(t) = Au_t \implies$$
$$y(t) = (A^* * u)(t) = \int_{-\infty}^{+\infty} dA^*(\theta)u(t-\theta)$$

- This convolution form :
  - enables a definition of the output y for irregular inputs (such as $u(t) = \delta(t)$),
  - is handy for Laplace analysis.

Laplace Transform Analysis

Consider $u \to y$ with $y(t) = Au_t$

- The **impulse response** (matrix) is A^* ,
- Its **transfer function** $A(s)$ is given by:

$$A(s) \triangleq \mathscr{L}(A^*)(s) = \int_{[-T,0]} \exp(s\theta)\, dA(\theta)$$

- For suitable inputs and values of $s \in \mathbb{C}$:

$$\mathscr{L}(y)(s) = A(s) \times \mathscr{L}(u)(s)$$

External Stability: Transfer Functions

- The transfer function of the I/O system is:

$$H(s) = C(s)(sI - A(s))^{-1}B(s) + D(s)$$

- Search for exponential solutions:

if $u(t) = u(s)e^{st}$, $y(t) = y(s)e^{st}$, $y(s) = H(s)u(s)$

- Its singularities are the **I/O poles**
- External stability:

iff all I/O poles s satisfy $\Re(s) < -\varepsilon < 0$.

Transfer Functions Examples

$$\left| \begin{array}{l} \dot{x}(t) = x(t-T) + u(t) \\ y(t) = x(t) \end{array} \right.$$

$$\left| \begin{array}{l} \dot{x}(t) = x(t) + u(t-T/2) \\ y(t) = x(t-T/2) \end{array} \right.$$

$$\left| \begin{array}{l} \dot{x}(t) = \int_{[-T,0]} x(t+\theta)\, d\theta + u(t) \\ y(t) = x(t) \end{array} \right.$$

From External Stability to Internal Stability

Take a FINITE-DIMENSIONAL linear system

- 1- get rid of all inputs and outputs,
- 2- add a new input in front of any integrator,
- 3- observe every state variable.

The new system is observable + commandable

$$\Longrightarrow$$

External Stability iff Internal Stability

Finite-Dimensional Systems Internal Stability

- Apply the process to a finite-dim. system:

$$\left|\begin{array}{l}\dot{x}(t) = Ax(t) + Bu(t)\\ y(t) = Cx(t) + Du(t)\end{array}\right. \longrightarrow \left|\begin{array}{l}\dot{x}(t) = Ax(t) + u(t)\\ y(t) = x(t)\end{array}\right.$$

$$H(s) = (sI - A)^{-1}$$

$$\begin{array}{rcl} s \in \mathbb{C} \text{ pole of } H(s) & \Longleftrightarrow & sI - A \text{ not invertible} \\ & \Longleftrightarrow & s \in \sigma(A) \end{array}$$

Conclusion - Internal Stability:

$$\forall s \in \sigma(A),\ \Re(s) < -\varepsilon < 0$$

Differential-Delay Systems Internal Stability

- Apply **formally** the process to a DDS:

$$\left|\begin{array}{l}\dot{x}(t) = Ax_t + Bu_t \\ y(t) = Cx_t + Du_t\end{array}\right. \longrightarrow \left|\begin{array}{l}\dot{x}(t) = Ax_t + u(t) \\ y(t) = x(t)\end{array}\right.$$

$$H(s) = (sI - A(s))^{-1}$$

$$\begin{array}{rcl} s \in \mathbb{C} \text{ pole of } H(s) & \Longleftrightarrow & sI - A(s) \text{ not invertible} \\ & \Longleftrightarrow & \det(sI - A(s)) = 0 \end{array}$$

Conclusion (valid !) - Internal Stability:

$$\forall\, s \in \mathbb{C}, \quad \det(sI - A(s)) = 0 \implies \Re(s) < -\varepsilon < 0$$

Differential-Delay Systems Internal Stability

- **Characteristic Matrix**

$$\Delta(s) = sI - A(s) = sI - \int_{[-T,0]} e^{\theta s} dA(\theta)$$

- **Characteristic Function / Equation:**

$$f(s) = \det \Delta(s) \quad / \quad f(s) = 0$$

- The roots of $f(s)$ are the **system poles**,
- **Spectral Exponential Stability Criterion**:

$$\sup \{\Re(s) \mid f(s) = 0\} < 0$$

Delay-Differential Algebraic Sys. Internal Stability

- For the class of delay systems:

$$\left| \begin{aligned} \dot{x}(t) &= Ax_t + By_t \\ y(t) &= Cx_t + Dy_t \end{aligned} \right.$$

the characteristic matrix is:

$$\Delta(s) = \begin{bmatrix} sI - A(s) & -B(s) \\ -C(s) & I - D(s) \end{bmatrix}$$

- The spectral stability criterion still works : the **spectral determined growth** condition holds.

Delay-Differential Systems External/Internal Poles

- $\Delta(s)$ is holomorphic, $\Delta(s)^{-1}$ merophormic :
  - finite or infinite number of poles,
  - but they are all isolated.
- There is a finite number of them such that

  $\Re(s) \geq M \ (\forall M)$ (delay-differential systems only)
- All external poles are internal poles

Internal Stability $\Rightarrow$ **External Stability**

Mikhailov Stability Criterion

- Consider $\dot{x}(t) = Ax_t, \; x(t) \in \mathbb{R}^n$
- The system is exponentially stable iff:

$$\forall \omega \in \mathbb{R}_+, \; f(i\omega) \neq 0$$

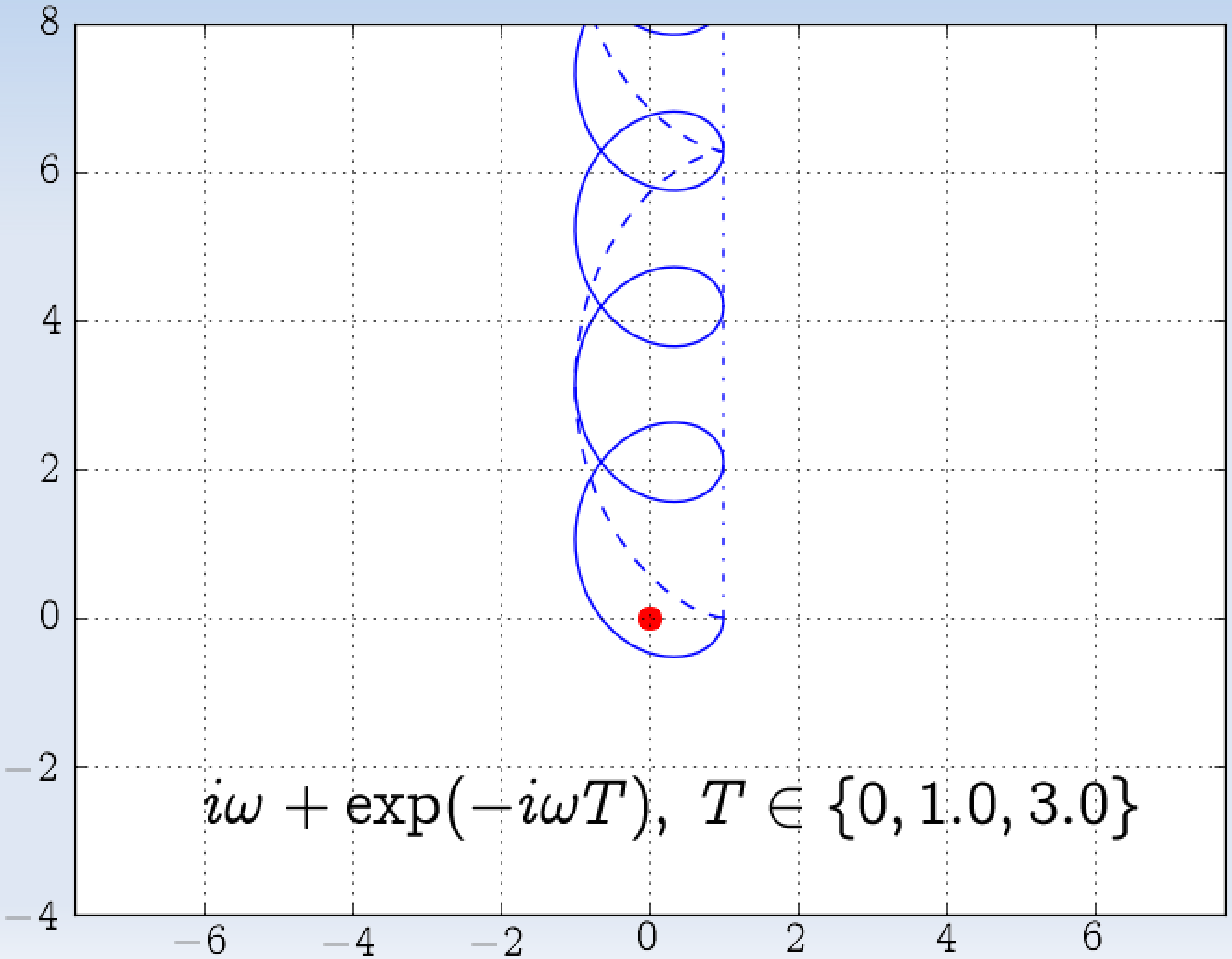
$$\Delta \arg f(i\omega)|_{\omega=0}^{+\infty} = n\frac{\pi}{2}$$

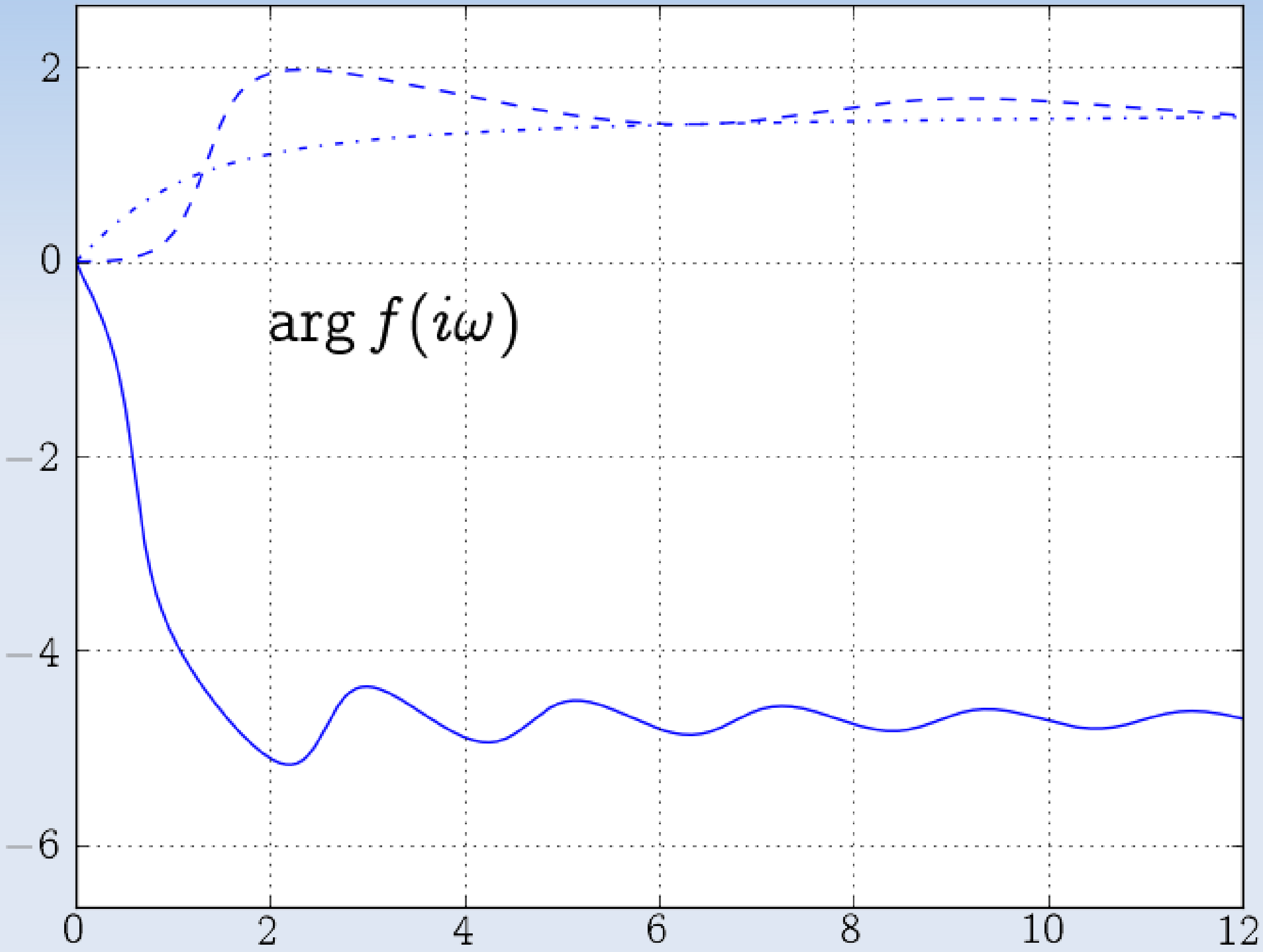
- N being the number of unstable poles

$$\Delta \arg f(i\omega)|_{\omega=0}^{+\infty} = n\frac{\pi}{2} - N\pi$$

Examples

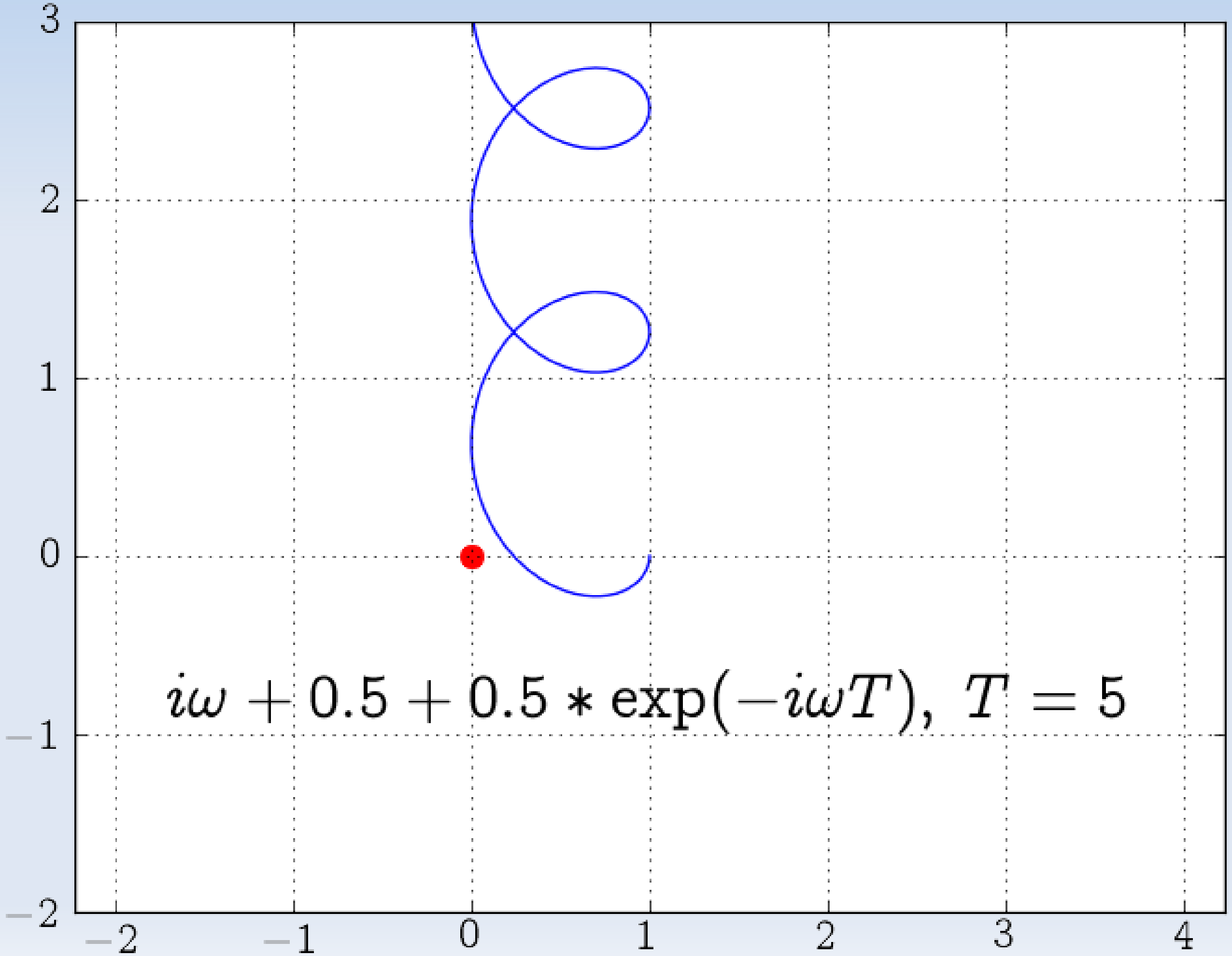
$$\dot{x}(t) = -x(t-T), \ x(t) \in \mathbb{R}$$



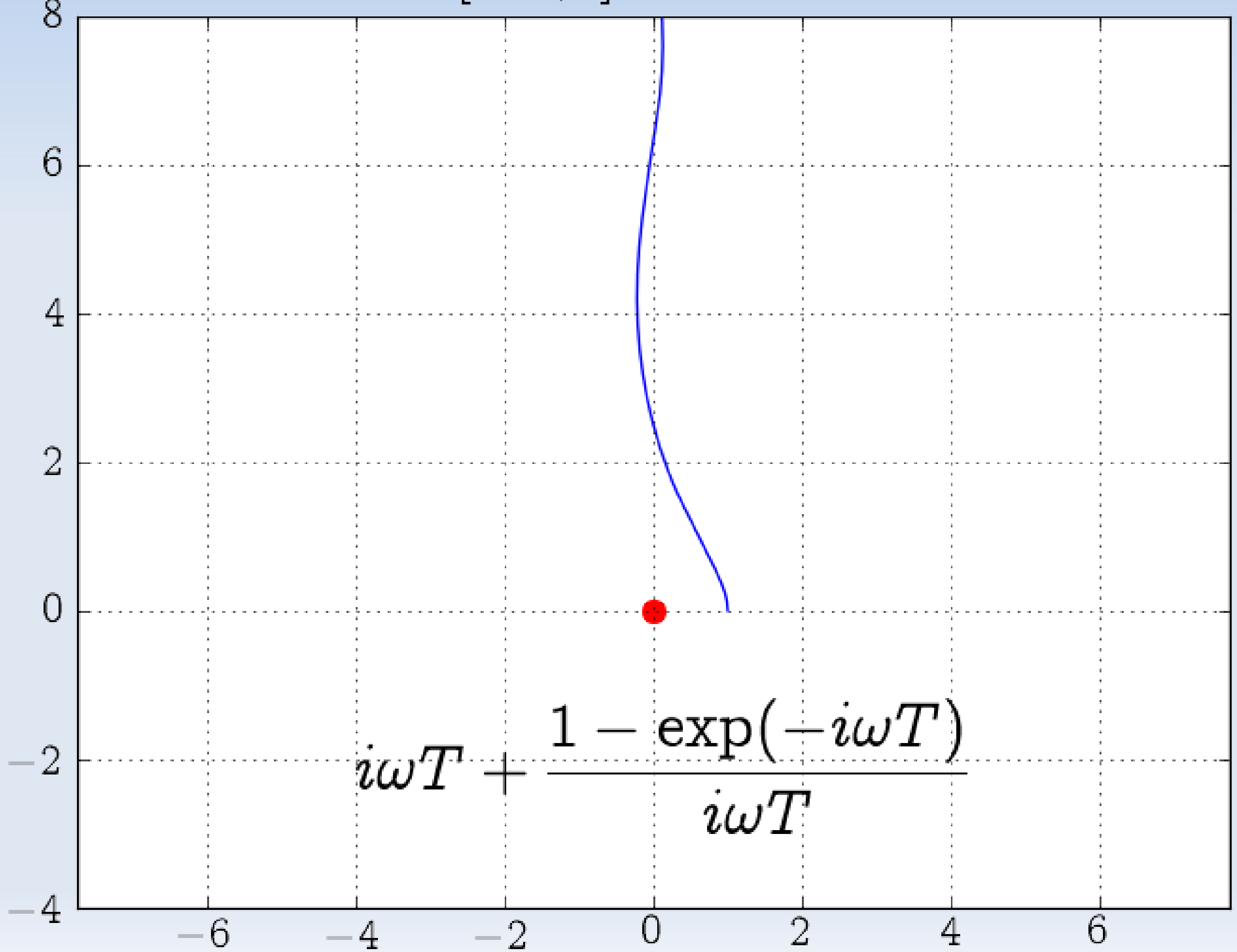


2
0
−2
−4
−6
0
2
4
6
8
10
12
arg $f(i\omega)$

$$\dot{x}(t) = -\frac{1}{2}[x(t) + x(t-T)],\ x(t) \in \mathbb{R}$$



$$T\dot{x}(t) = -\frac{1}{T}\int_{[-T,0]} x(t+\theta)\,d\theta,\ \ x(t)\in\mathbb{R}$$



Stability for every delay

$$\dot{x}(t) = Ax(t) + Bx(t-T)$$

- If the system exp. stable for T=0,
- If $\sigma(A) \cap \{i\omega,\, \omega \in \mathbb{R}_+^*\} = \varnothing$,
- Let $H(i\omega) = (i\omega - A)^{-1}B$,

The system is exp. stable **for any delay** iff:

$$\forall\, \omega \in \mathbb{R}_+,\ \rho(H(i\omega)) < 1$$

$$\text{with}\quad \rho(A) = \sup\{|\lambda|,\ \lambda \in \sigma(A)\}$$

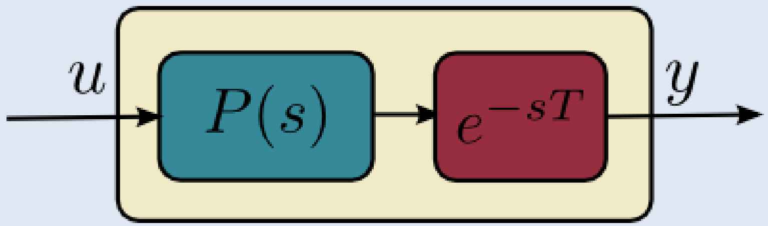
Delay Systems: Controller Design

Control of Dead-Time Systems : from the Smith predictor to Observer-Predictor structures.

Dead-Time Systems

The most common delay system structure:

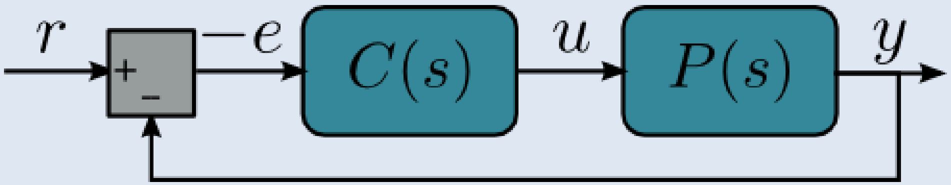
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t-T) \end{cases} \qquad u(t) \in \mathbb{R},\ \ y(t) \in \mathbb{R}$$



$$P(s) = C(sI - A)^{-1}B$$

Smith Predictor
Step I: delay-free controller

- **Without delay**, get a 1-d.o.f. controller with:
  - Internal closed-loop exp. stability,
  - a suitable $r \to y$ behavior (t.f. $H(s)$).



$$H(s) = (I + PC(s))^{-1} PC(s)$$

$$\left| \begin{aligned} \dot{z} &= Ez + F(-e) \\ u &= Gz + H(-e) \end{aligned} \right. \qquad C(s) = G(sI - E)^{-1} F + H$$

Smith Predictor
Step I: delay-free controller

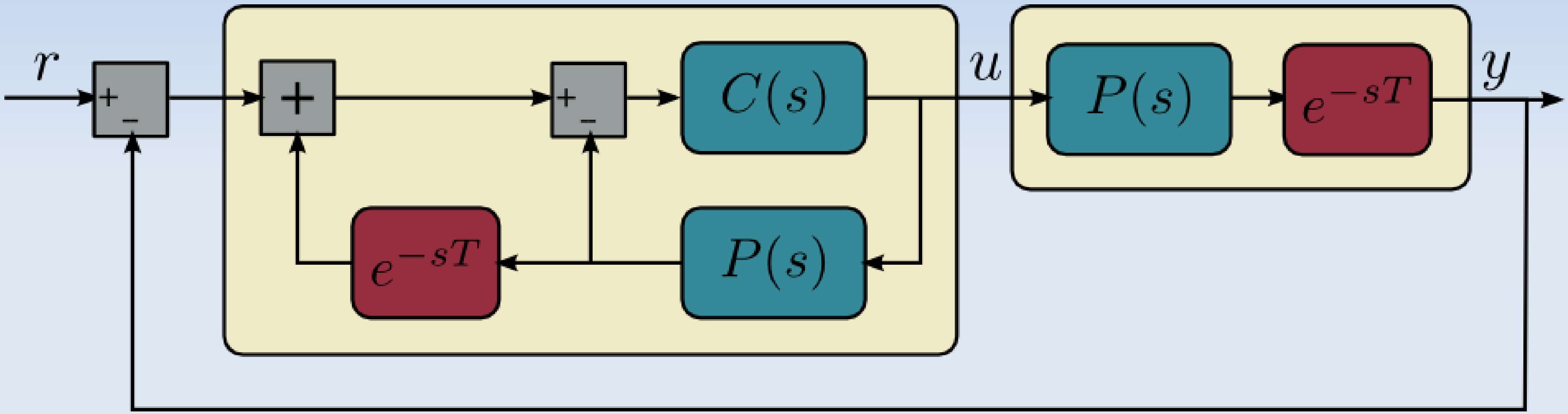
Closed-Loop State-Space Dynamics:

x plant state, z controller state

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix}(t) = M \begin{bmatrix} x \\ z \end{bmatrix}(t)$$

$$M = \begin{bmatrix} A - BHC & BG \\ -FC & E \end{bmatrix}$$

Smith Predictor
Step II: build the controller

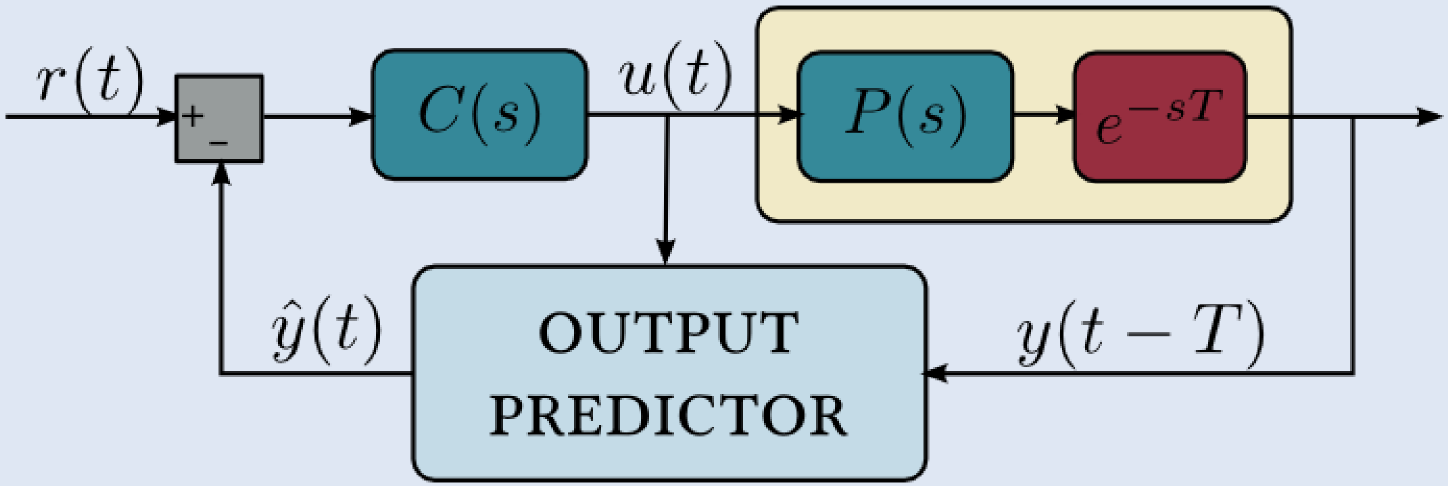


There is no step III !

Smith Controller Design

- How do you think of such a design ?

Adopt the general (sensible) structure:

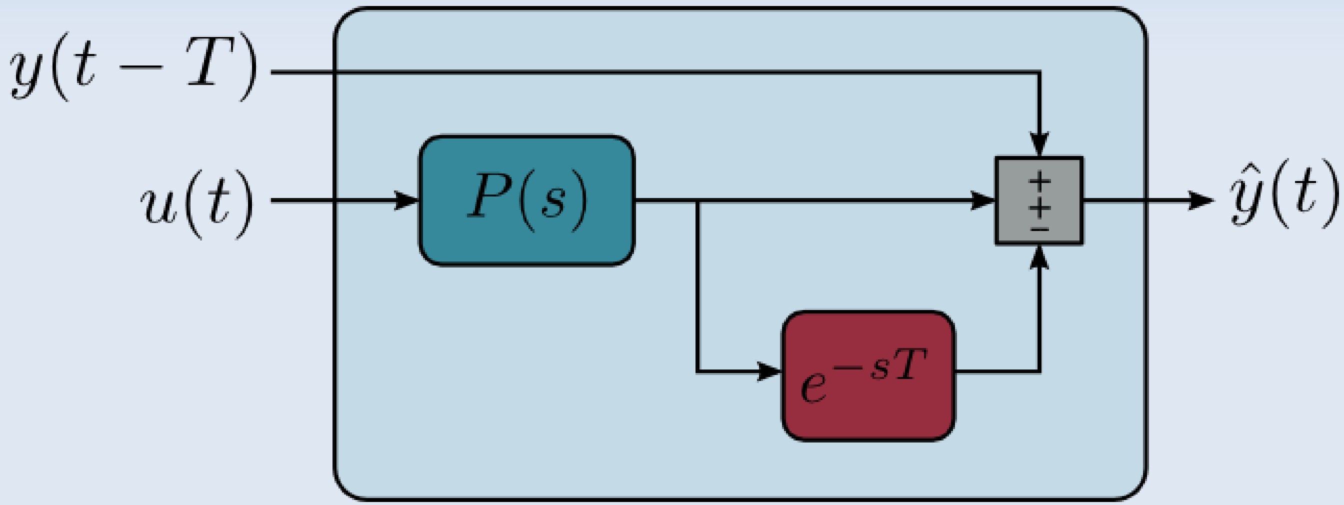


Output Predictor Design

- Try first an 'open loop predictor'.
  - what is the issue ?
- Close the loop
  - ... but preserve the same I/O behavior !
- Among predictor designs
  - ... select the one that creates a one-degree of freedom controller.

Smith (Output) Predictor

- Builds an output estimate from the delayed output measure and applied control:



Smith Predictor Analysis

- I/O behavior:

$$H(s) = (I + PC(s))^{-1} PC(s) e^{-sT}$$

that is the delay-free behavior, delayed.

- What about internal stability ?

Smith Predictor State-Space Model

- Denote

$$\left|\begin{array}{l} x \text{ plant state,} \\ z \text{ controller state} \\ e \text{ prediction error} \end{array}\right.$$

- Closed-loop system dynamics structure:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{e} \end{bmatrix}(t) = \mathcal{A}\begin{bmatrix} x \\ z \\ e \end{bmatrix}(t) + \mathcal{A}_d\begin{bmatrix} x \\ z \\ e \end{bmatrix}(t-T)$$

Smith Predictor
Closed Loop Dynamics

$$\mathcal{A} = \begin{bmatrix} A - BHC & BG & -BHC \\ -FC & E & -FC \\ 0 & 0 & A \end{bmatrix}$$

$$\mathcal{A}_d = \begin{bmatrix} 0 & 0 & BHC \\ 0 & 0 & FC \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Delta(s) = sI - \mathcal{A} - \mathcal{A}_d \exp(-sT) = \begin{bmatrix} sI - M & ? \\ 0 & sI - A \end{bmatrix}$$

Smith Predictor Characteristic Function

$$f(s) = \det(sI - M) \times \det(sI - A)$$

poles of the closed-loop system

=

poles of the delay-free closed-loop system

+

poles of the plant open loop

Smith Predictor Conclusions

- Nice I/O behavior,
- Internal exp. stability:
  - iff the plant is already exp. stable

(rk: potential unstability was visible from I/O analysis)

What about a STATE predictor strategy ?

Exact State Predictor I

Assume that for $t \geq 0$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

then for $t_1 \geq t_0$

$$x(t_1) = e^{A(t_1 - t_0)} x(t_0) + \int_{t_0}^{t_1} e^{A(t_1 - t)} Bu(t)\, dt$$

so for any $t \leq T$

$$x(t) = e^{AT} x(t - T) + \int_{t-T}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau$$

Exact State Predictor

- The predictor can be realized without using distributed delays as :

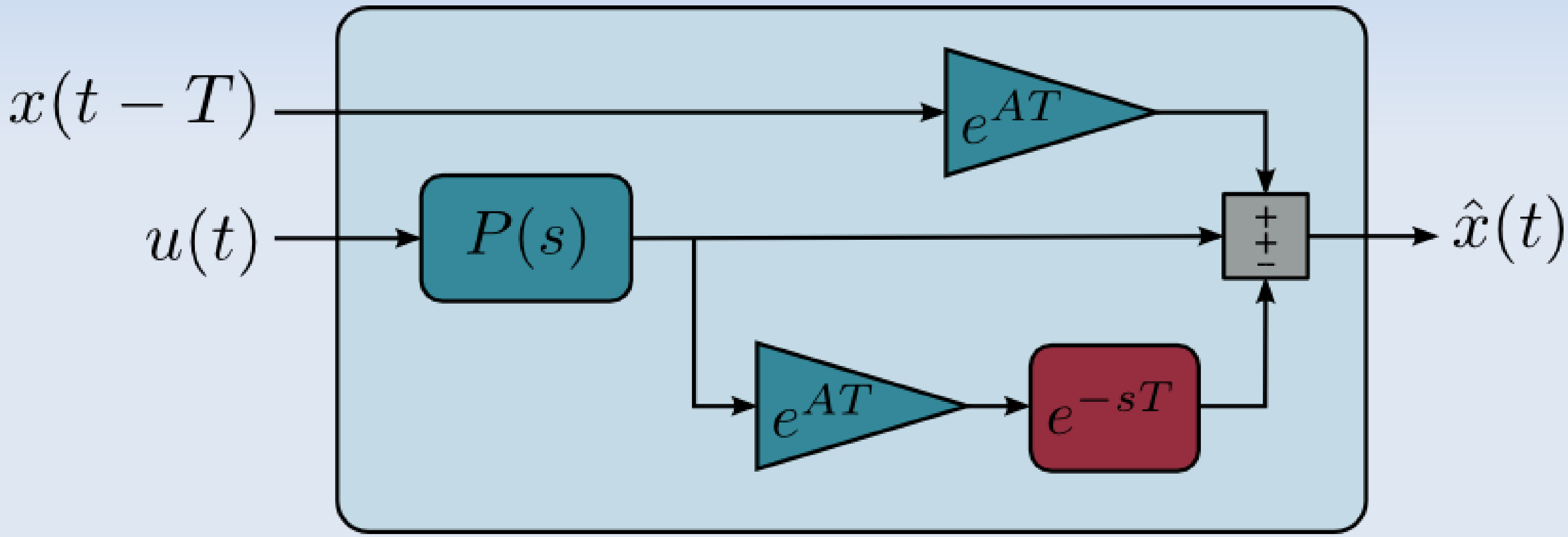
$$x(t) = e^{AT}x(t-\tau) + \xi(t) - e^{AT}\xi(t-\tau)$$

$$\text{where} \left| \begin{array}{l} \dot{\xi} = A\xi + Bu \\ \xi(0) = 0 \end{array} \right.$$

- This is a **bad idea** ! (more on this subject later)

Exact State Predictor

Still, the structure is interesting:

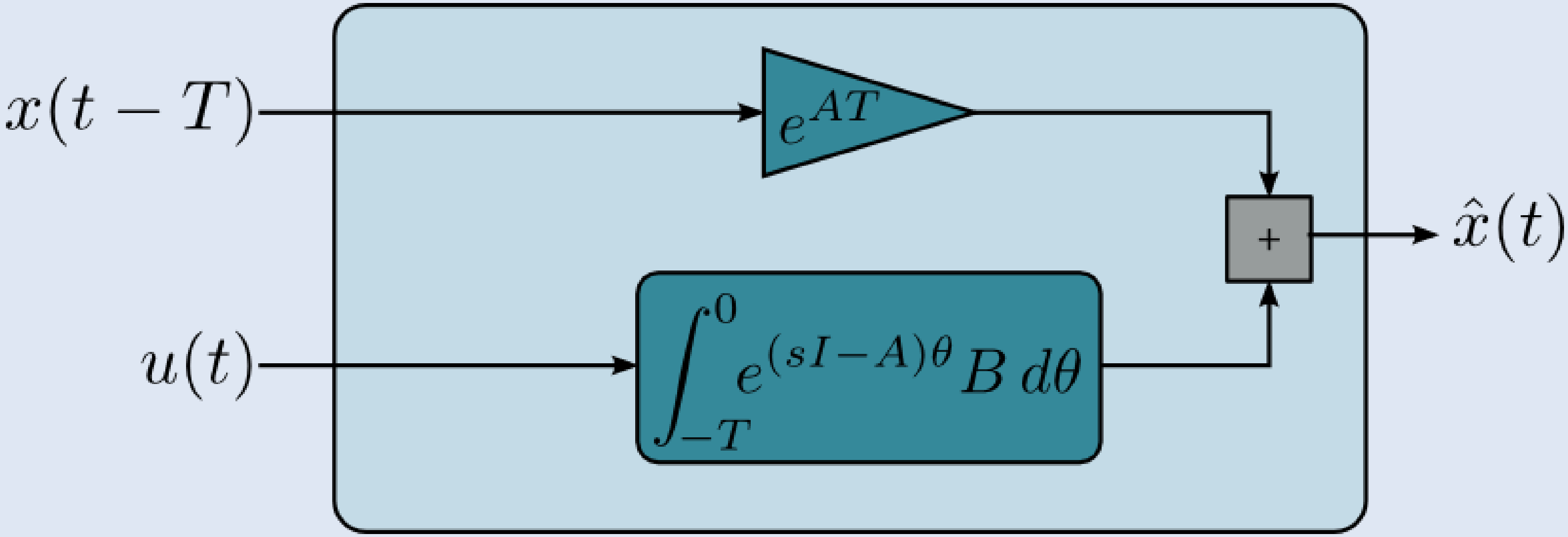


(compare with Smith predictor)

Exact State Predictor II

- define the state estimator $\hat{x}(t)$ for $t \geq 0$ by:

$$\hat{x}(t) = e^{AT}x(t-T) + \int_{[-T,0]} \left[e^{-A\theta}B d\theta\right] u_t(\theta)$$



- the estimation is **exact** for $t \geq T$

Finite-Spectrum Assignment Controller Design

- Step 1: find a matrix gain K such that

$$\left|\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ u(t) &= -Kx(t)\end{aligned}\right.$$

is asymptotically stable.

- Step 2: as $x(t)$ is not available, use instead

$$u(t) = -K\hat{x}(t)$$

FSA Stability Analysis

- State-space model:

$$\begin{cases} \dot{x}(t) = Ax(t) - BK\hat{x}(t) \\ \hat{x}(t) = e^{AT}x(t-T) - \int_{[-T,0]} \left[e^{-A\theta}BK\,d\theta\right]\hat{x}_t(\theta) \end{cases}$$

- Laplace analysis - matrix $\Delta(s)$

$$\left[\begin{array}{c|c} sI - A & BK \\ \hline -e^{-(sI-A)T} & I + [sI - A]^{-1}(I - e^{-(sI-A)T})BK \end{array}\right]$$

$$\text{charac. func.: } f(s) = \det \Delta(s)$$

Set $L(s) = [sI - A]^{-1}[I - e^{(sI-A)T}]$

then

$$\Delta(s)\left[\begin{array}{c|c} I & 0 \\ \hline I & I \end{array}\right] = \mathcal{P}(s)\left[\begin{array}{c|c} sI - A + BK & 0 \\ \hline 0 & I \end{array}\right]$$

with $\mathcal{P}(s) = \left[\begin{array}{c|c} I & BK \\ \hline L(s) & I + L(s)BK \end{array}\right]$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline L(s) & I \end{array}\right]\left[\begin{array}{c|c} I & BK \\ \hline 0 & I \end{array}\right]$$

FSA Conclusions

$$f(s) = \det \Delta(s) = \det(sI - A + BK)$$

- finite number of poles
- internal exp. stability:

delayed loop with FSA poles

=

delay-free loop with state feedback poles.

- Exercice: consider the FSA controller but with a predictor implemented as a Smith structure (no distributed delay).
- Show without computations that the open-loop system poles are still poles of the closed loop (therefore the closed loop is internally exp. stable iff the original system is)

FSA with Observers

- What if $x(t-T)$ is not available but only $y(t)$?

$$\left| \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t-T) \end{aligned} \right.$$

- Assume that (A, C) is observable. We can build an **observer** to get an estimate $\tilde{x}(t)$ of the delayed state $x(t-T)$.

Observer Design

- Pick up L such that $A - LC$ is stable.
- Consider

$$\left| \begin{array}{l} \dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t-T) - L(\tilde{y}(t) - y(t)) \\ \tilde{y}(t) = C\tilde{x}(t) \end{array} \right.$$

- The error $e(t) = \tilde{x}(t) - x(t-T)$ satisfies :

$$\dot{e}(t) = (A - LC)e(t)$$

therefore $e(t) \to 0 \text{ when } t \to +\infty$

Integration

- Adapt the FSA control law :

chain the observer and the predictor

$$\left| \begin{array}{l} u(t) = -K\hat{x}(t) \\ \hat{x}(t) = e^{AT}\tilde{x}(t) - \int_{[-T,0]} [e^{-A\theta} BKd\theta]\,\hat{x}_t(\theta) \end{array} \right.$$

- Is the full construct internally exp. stable ?

FSA with Observers : Stability Analysis

- Consider the dynamics of $(x(t), \hat{x}(t), e(t))$,
- The associated characteristic matrix is :

$$\mathcal{M}(s) = \left[\begin{array}{c|c} \Delta(s) & ? \\ \hline 0 & sI - A + LC \end{array}\right]$$

- The characteristic equation is:

$$\det(\mathcal{M}(s)) = \det(sI - A + BK) \times \det(sI - A + LC)$$

$$\Rightarrow \quad \text{internal exp. stability}$$

Example

- Consider the scalar equation :

$$\ddot{y}(t) = u(t - T)$$

- Design a predictor-observer-controller that stabilizes exponentially this system,
- More precisely, locate all closed-loop poles at $s = -1$.

- State-space model:

$$x(t) = \begin{bmatrix} y(t+T) \\ \dot{y}(t+T) \end{bmatrix} \in \mathbb{R}^2$$

$$\left| \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t-T) \end{array} \right.$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Finite-Dimensional Observer/Controller Design

- The gain matrices :

$$K = \begin{bmatrix} 1 & 2 \end{bmatrix}, \; L = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are such that :

$$\sigma(A - BK) = \{-1, -1\}$$

$$\sigma(A - LC) = \{-1, -1\}$$

$\Longrightarrow$ poles of the closed-loop system:

$$\{-1, -1, -1, -1\}$$

- Predictor + Observer + Controller structure:

$$\left|\begin{array}{l}\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t-T) - L(C\tilde{x}(t) - y(t)) \\ u(t) = -K\hat{x}(t) \\ \hat{x}(t) = e^{AT}\tilde{x}(t) - \int_{[-T,0]} \left[e^{-A\theta}BK\,d\theta\right]\hat{x}_t(\theta)\end{array}\right.$$

$$\text{for } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e^{At} = \sum_{n=0}^{+\infty} \frac{1}{n!}(tA)^n = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$
\left|\begin{array}{rcl}
\hat{x}_1(t) & = & \tilde{x}_1(t) + T\tilde{x}_2(t) + \int_{[-T,0]} \theta(\hat{x}_1(t+\theta) + 2\tilde{x}_2(t+\theta))d\theta \\
\hat{x}_2(t) & = & \tilde{x}_2(t) - \int_{[-T,0]} \hat{x}_1(t+\theta) + 2\tilde{x}_2(t+\theta)d\theta \\
\frac{d}{dt}\tilde{x}_1(t) & = & \tilde{x}_2(t) - 2\tilde{x}_1(t) + 2y(t) \\
\frac{d}{dt}\tilde{x}_2(t) & = & -\hat{x}_1(t) - 2\hat{x}_2(t) - \tilde{x}(t) + y(t) \\
u(t) & = & -\hat{x}_1(t) - 2\hat{x}_2(t)
\end{array}\right.
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Short Reference List

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