MATHEMATIQUES ET SYSTEMES, 10/10/2013

Delay Equations

A Case for Algebro-Differential Systems

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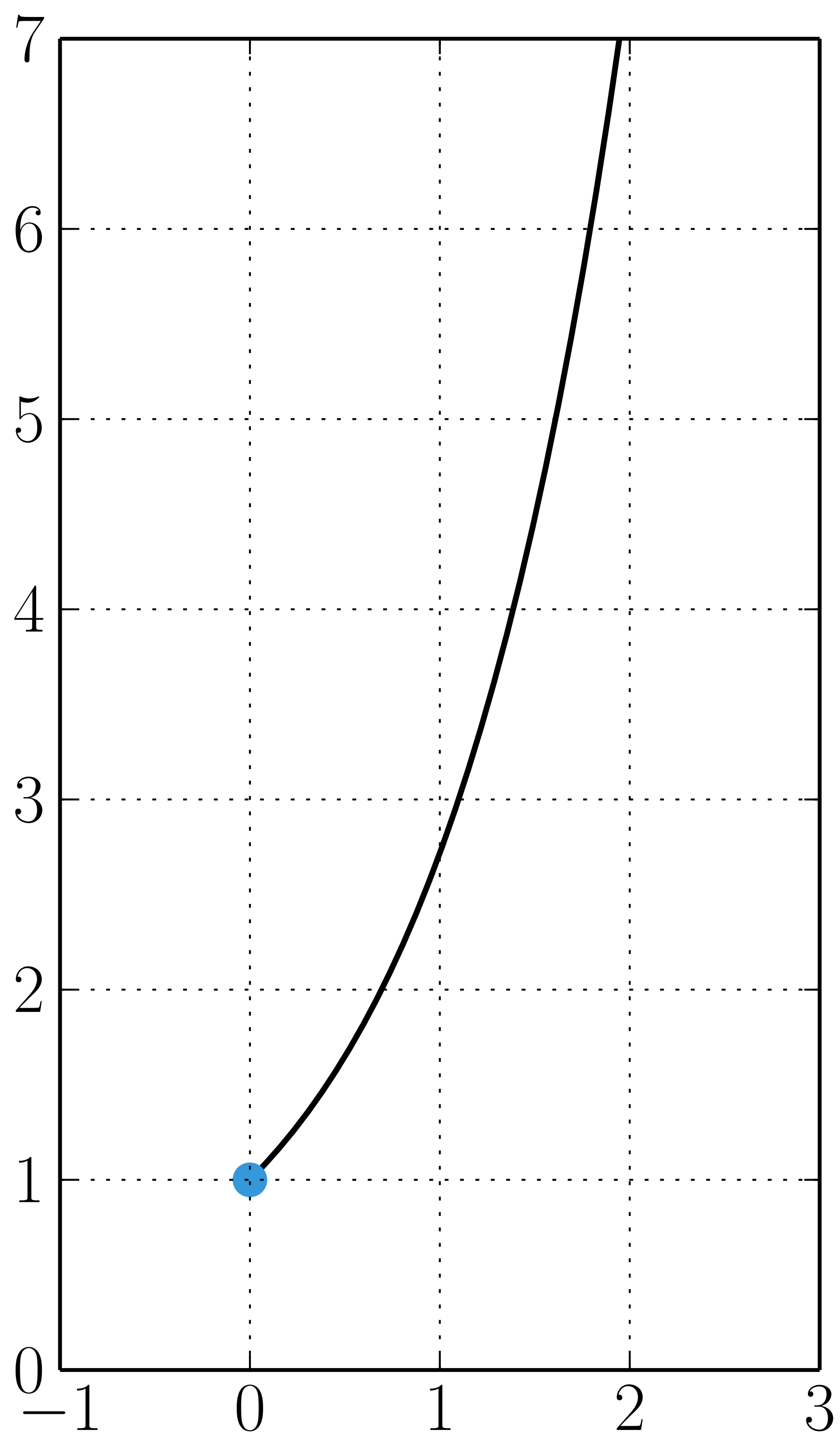
Differential Equations

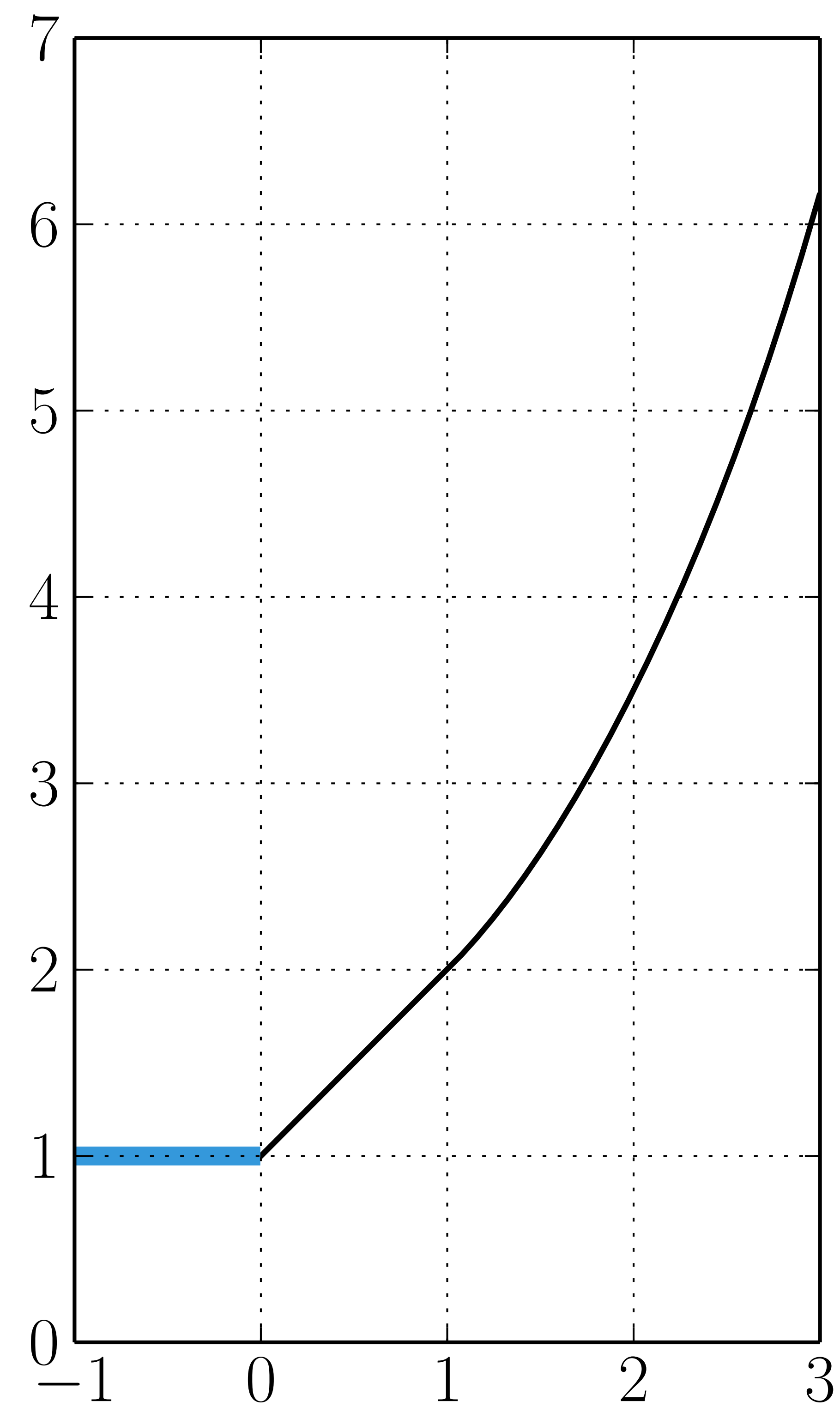
ODE — Ordinary

$$\dot{x}(t) = x(t)$$

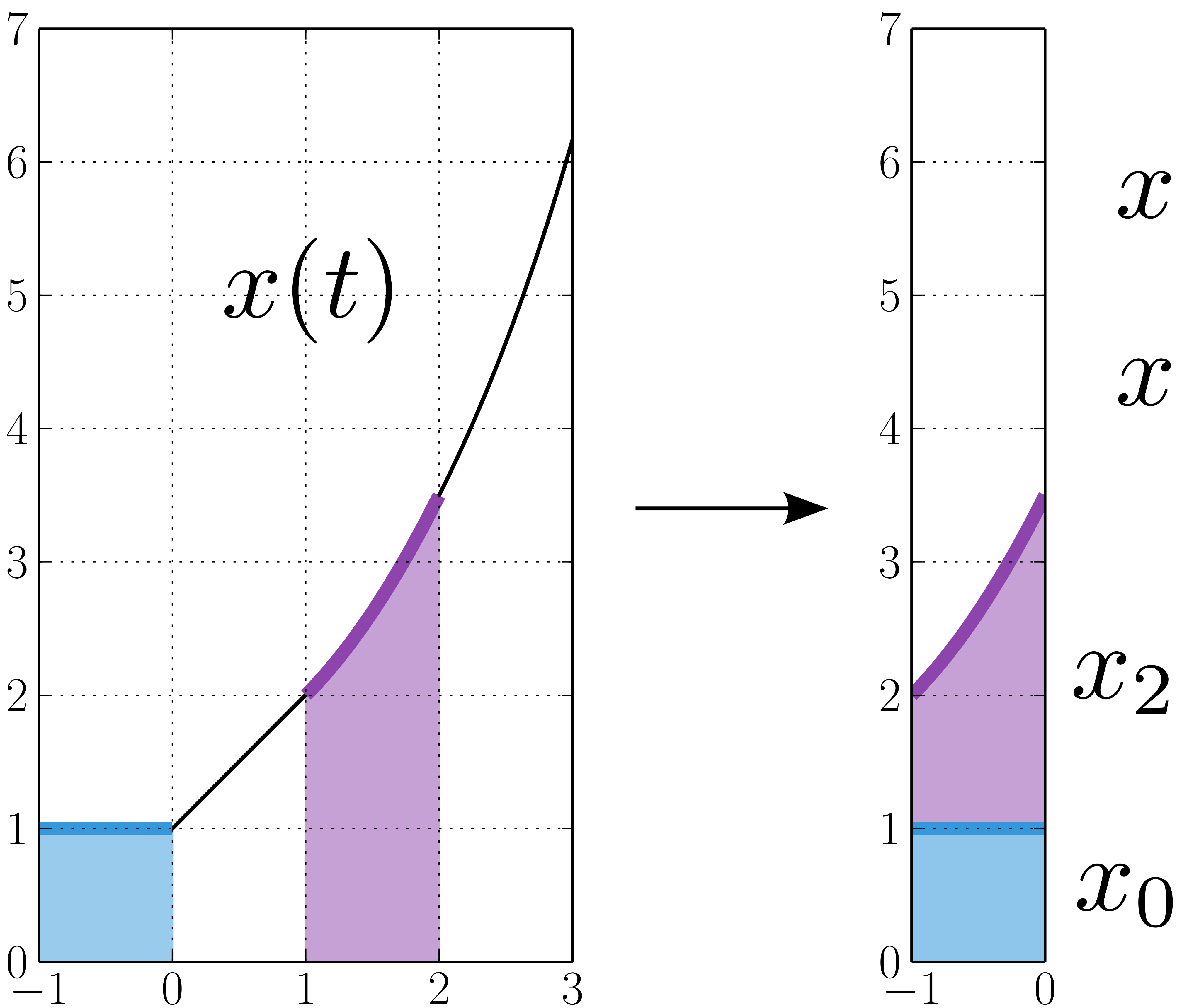


DDE — Delay

$$\dot{x}(t) = x(t-1)$$



DDE — State Space



$$x_t : [-\tau, 0] \to \mathbb{R}^n$$

$$x_t(\theta) = x(t + \theta)$$

Discrete/Distributed Delay

$$\sum_i a_i x(t - \tau_i) = A x_t$$

$$A\phi = \sum_i a_i \phi(-\tau_i)$$

$$\int_{t-\tau}^{t} a(\theta - t) x(\theta)\, d\theta = A x_t$$

$$A\phi = \int_{-\tau}^{0} a(\theta)\phi(\theta)\, d\theta$$

Continuous Framework

State-Space

$$X^j = C^0([-\tau, 0], \mathbb{R}^j)$$

Delay Operator

$$A \in \mathcal{L}(X^j, \mathbb{R}^i)$$

Functional-Differential Equation

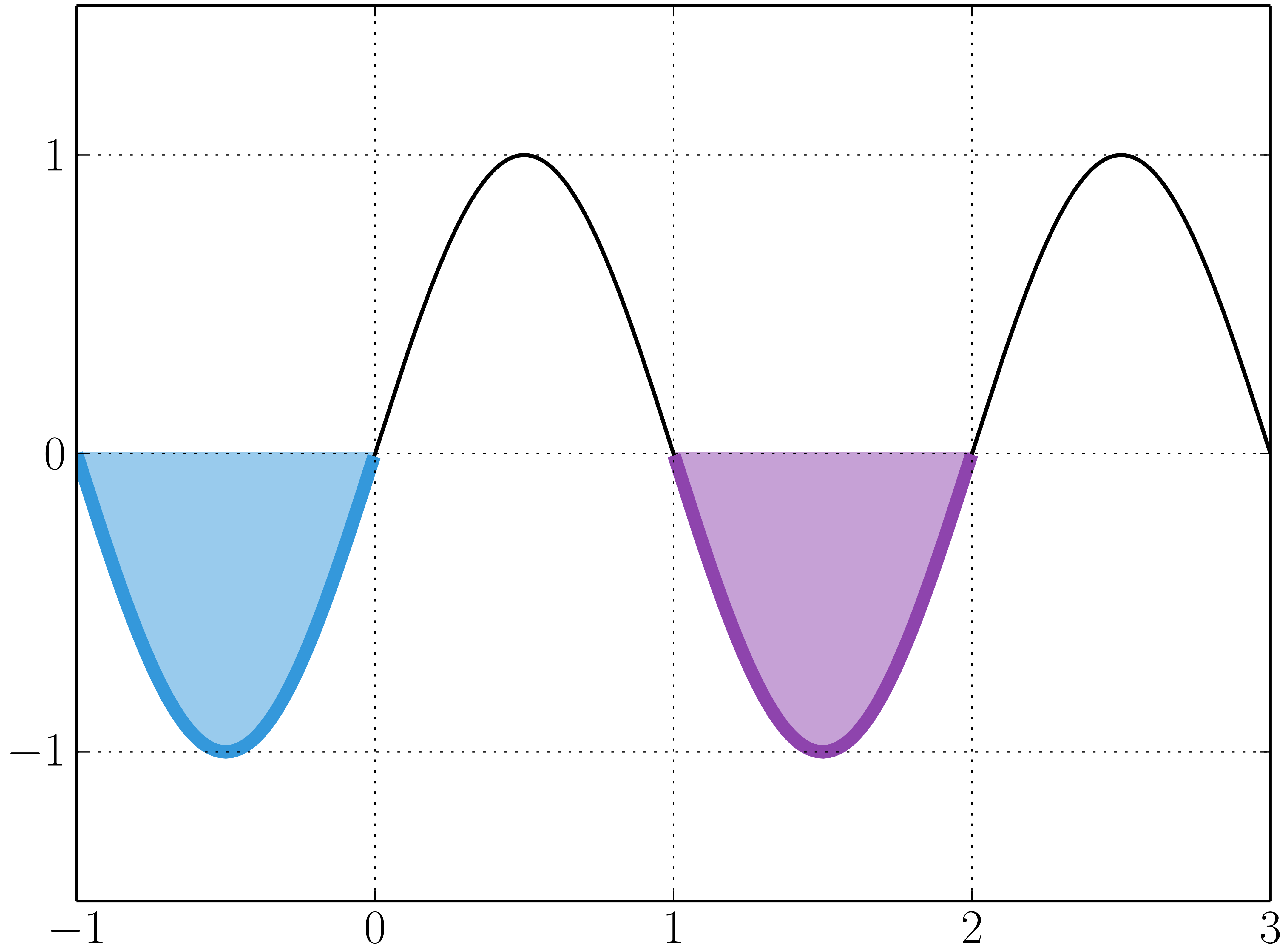
$$\dot{x}(t) = Ax_t$$

$$A \in \mathcal{L}(X^n, \mathbb{R}^n)$$

Delay Algebraic Equations

a.k.a. Difference Equations

$$y(t) = -y(t-1) \quad \text{(or } y_t = y_{t-1}\text{)}$$



DDAE

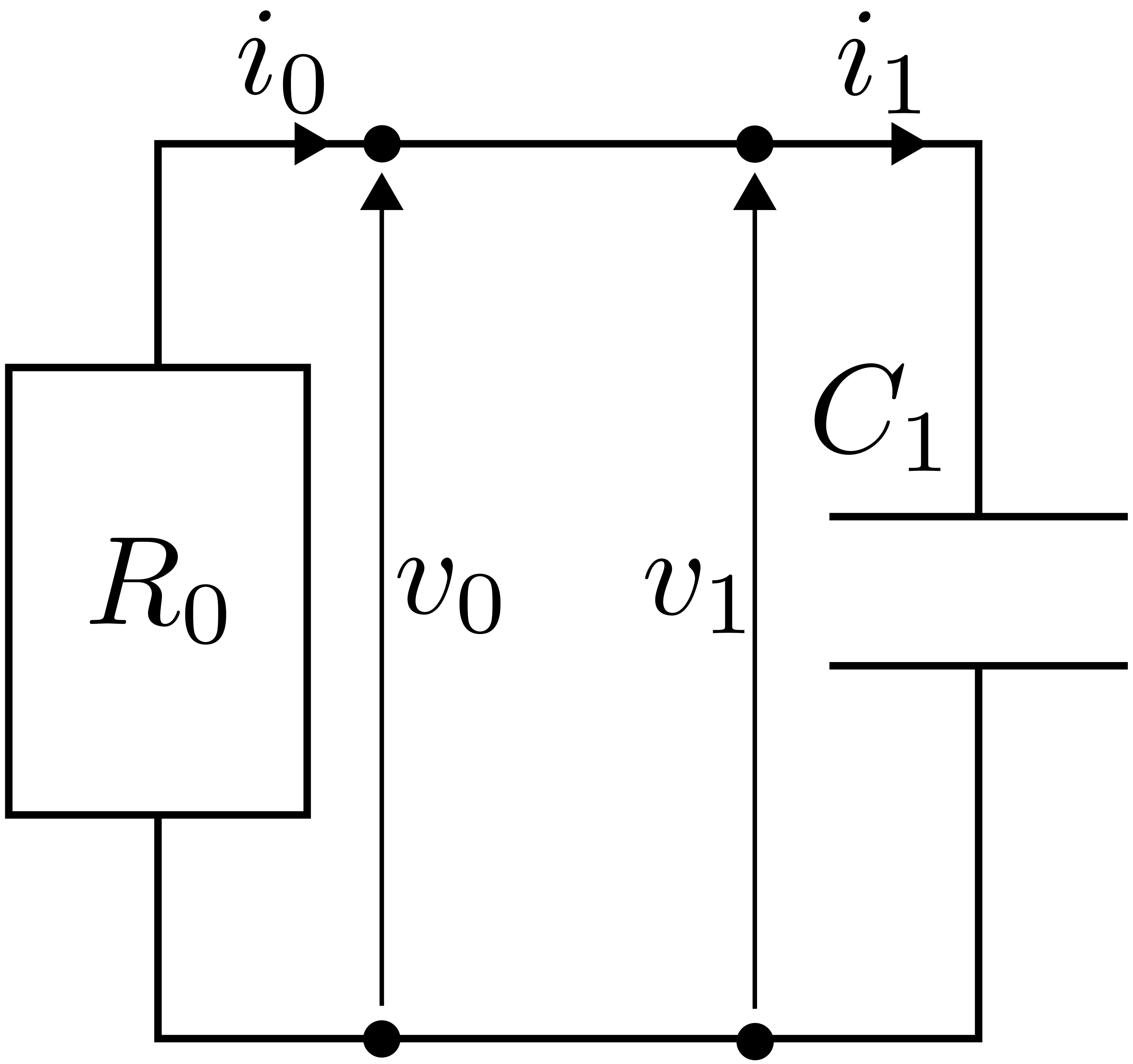
Delay-Differential Algebraic Equations

$$\left| \begin{array}{l} \dot{x}(t) = Ax_t + By_t \\ y(t) = Cx_t + Dy_t \end{array} \right.$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(X^{n+m}, \mathbb{R}^{n+m})$$

Modeling & Physics

RLC Circuits



$$v_0 = -R_0 i_0$$

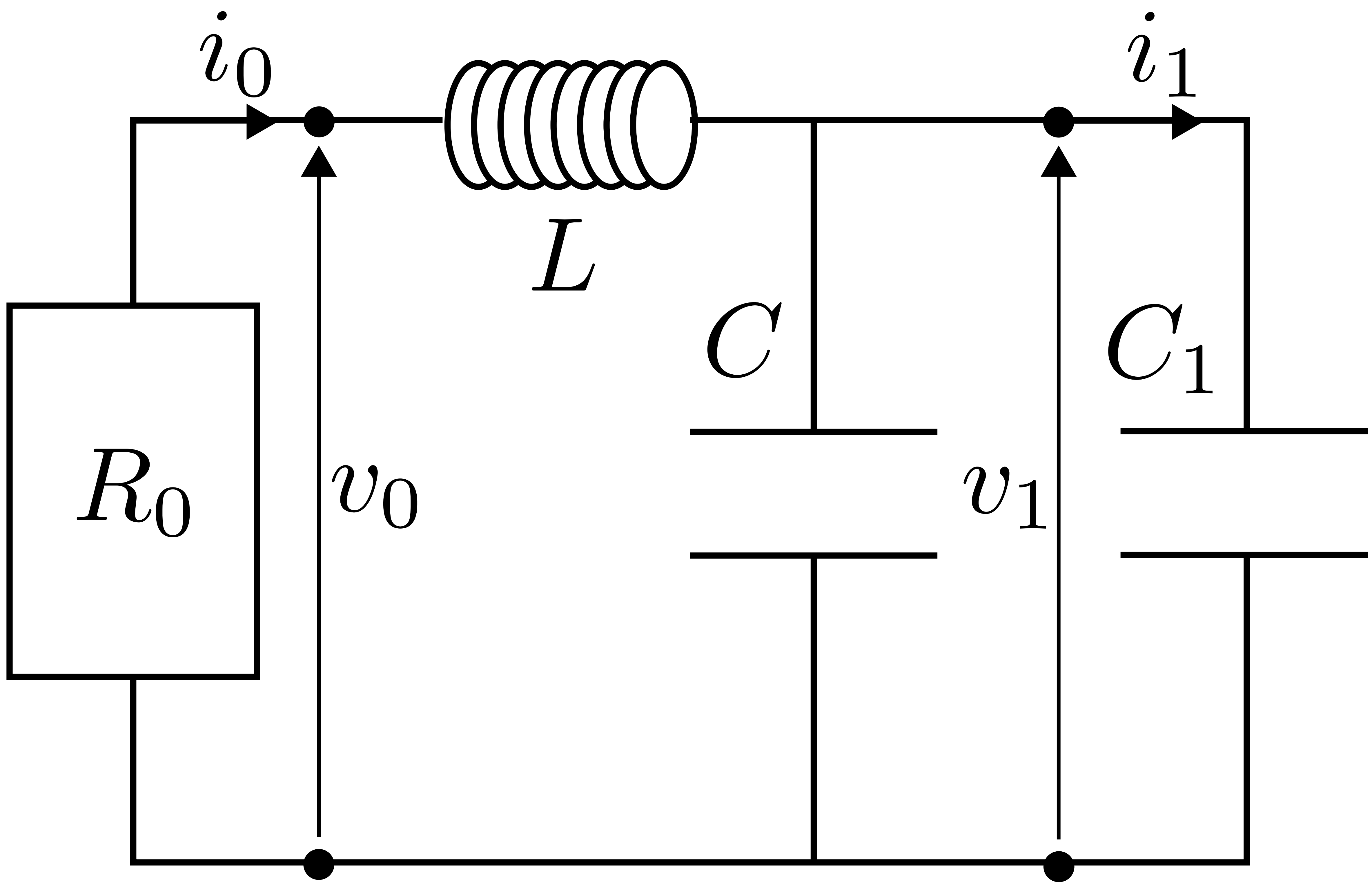
$$i_1 = C_1 \dot{v}_1$$

$$i_0 = i_1$$

$$v_0 = v_1$$

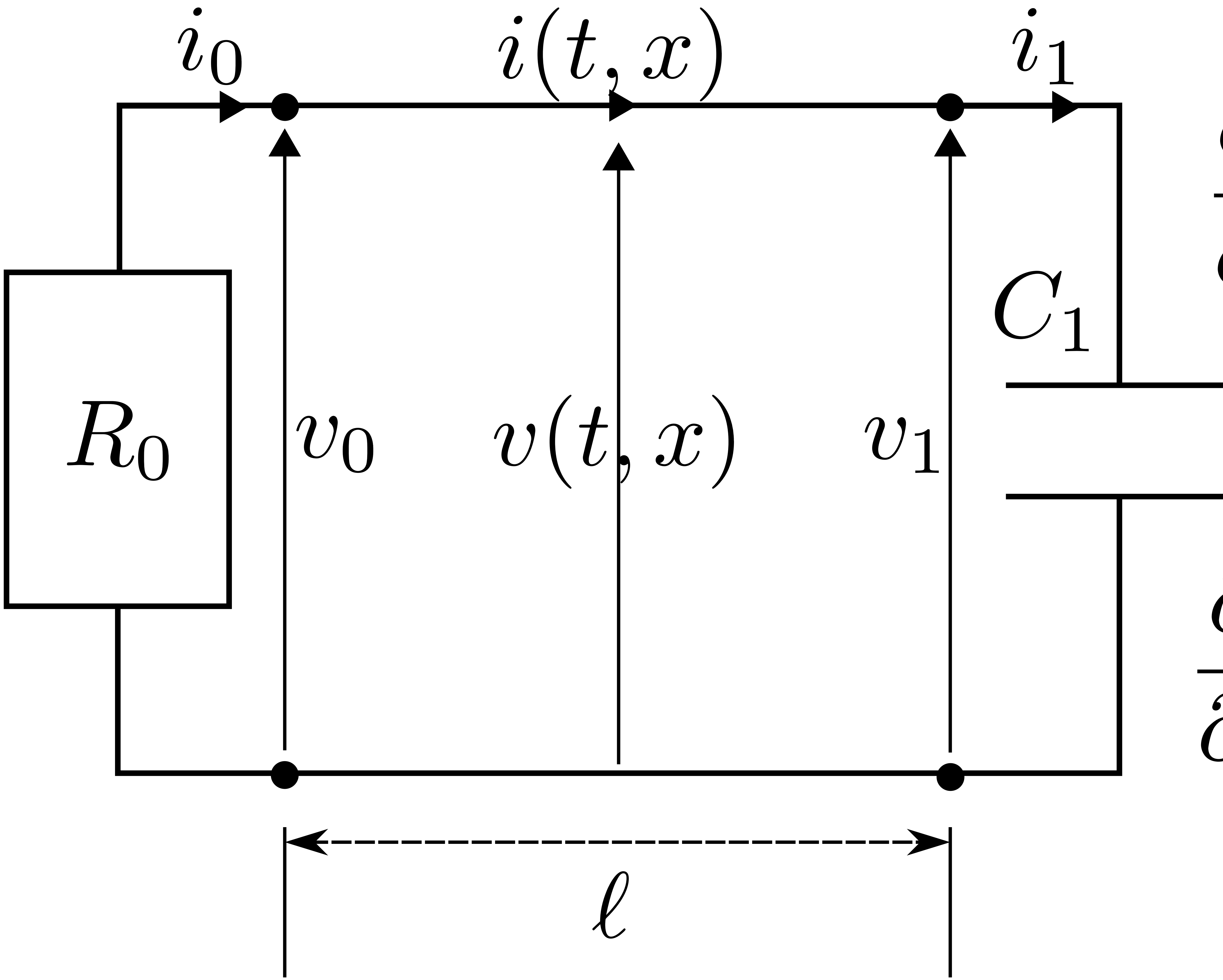
Losless Transmission Line

$$v_1 - v_0 = -L di_0/dt$$



$$i_1 - i_0 = -C dv_1/dt$$

Losless Transmission



$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}$$

Wave Equation

$$\frac{\partial^2 v}{\partial t^2}(t,x) = c^2 \frac{\partial^2 v}{\partial x^2}(t,x) \qquad c = \frac{1}{\sqrt{LC}}$$

$$\frac{\partial^2 i}{\partial t^2}(t,x) = c^2 \frac{\partial^2 i}{\partial x^2}(t,x) \qquad Z = \sqrt{\frac{L}{C}}$$

$$v(t,x) = v_+(t - x/c) + v_-(t + (x - \ell)/c)$$

$$Zi(t,x) = v_+(t - x/c) - v_-(t + (x - \ell)/c)$$

Wave Equation
Nodal Values

Let $\tau = \ell/c$

$$v_0(t) = v_+(t) + v_-(t-\tau)$$

$$Zi_0(t) = v_+(t) - v_-(t-\tau)$$

$$v_1(t) = v_+(t-\tau) + v_-(t)$$

$$Zi_1(t) = v_+(t-\tau) - v_-(t)$$

Dynamics

$$\frac{dv_1}{dt}(t) = \frac{1}{C_1 Z}(2v_+(t-\tau) - v_1(t))$$

$$v_-(t) = v_1(t) - v_+(t-\tau)$$

$$v_+(t) = \kappa v_-(t-\tau) \qquad \kappa = \frac{R_0 - Z}{R_0 + Z}$$

Dynamics

Select $x(t) = [v_1(t)]$, $y(t) = \begin{bmatrix} v_+(t) \\ v_-(t) \end{bmatrix}$.

$$\left| \begin{aligned} \dot{x}(t) &= Ax(t) + By(t-\tau) \\ y(t) &= Cx(t) + Dy(t-\tau) \end{aligned} \right.$$

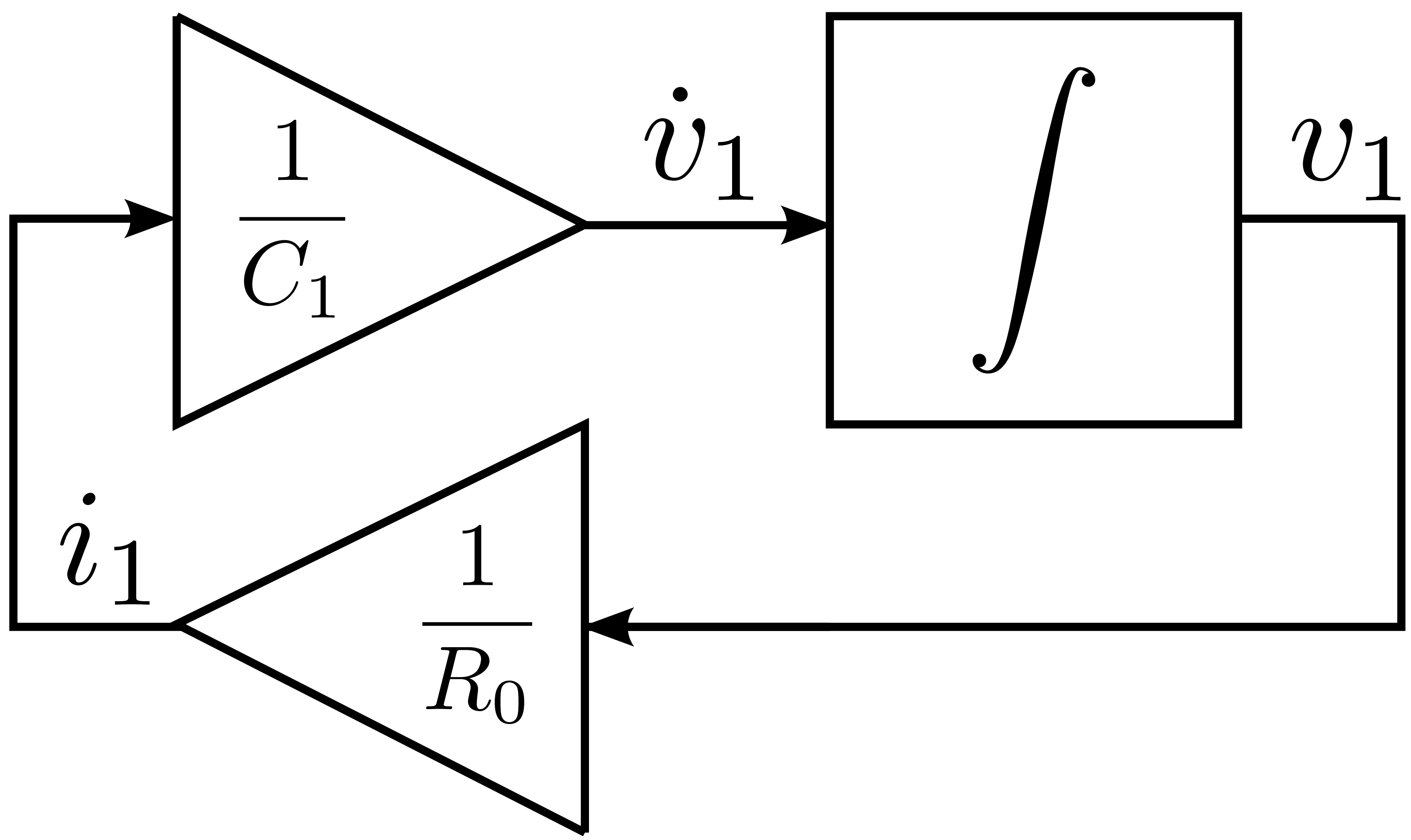
Networks of T.L. + RLC elements.
(Brayton, 1968)

Block Diagrams
&
Well-Posedness

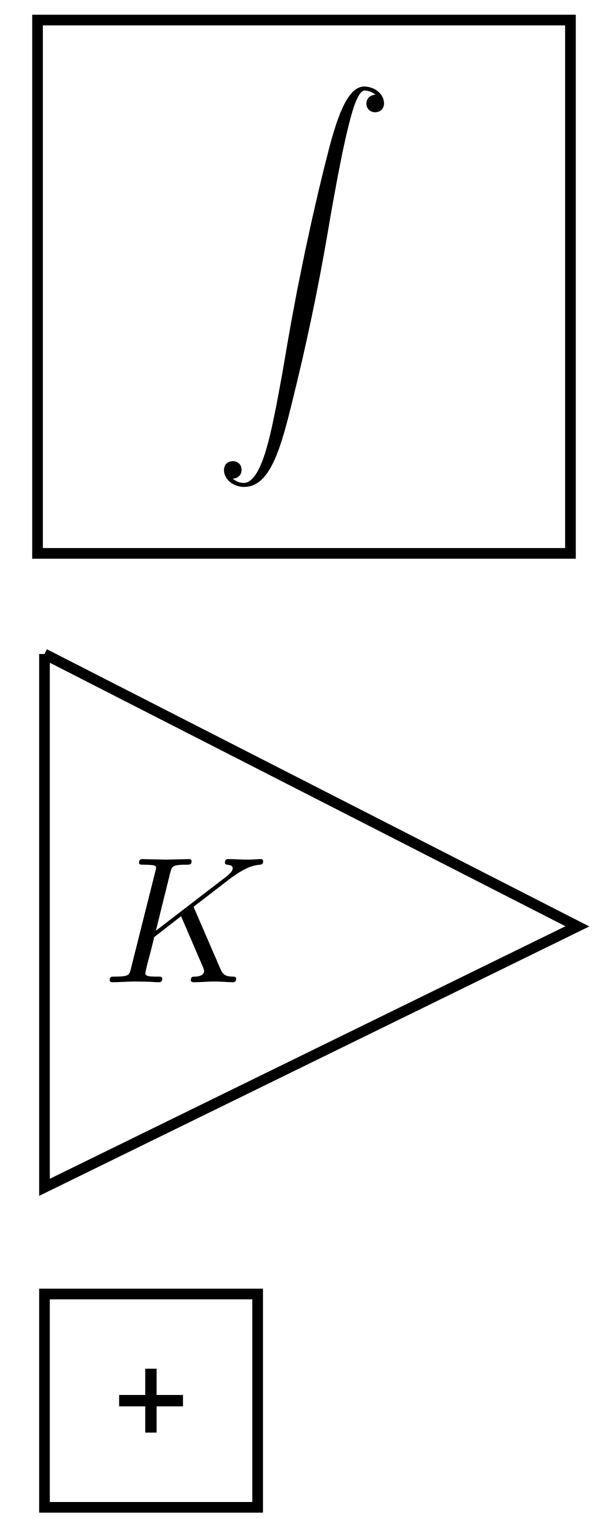
Block-Diagrams

ODEs — Linear Time-Invariant

RC Circuit

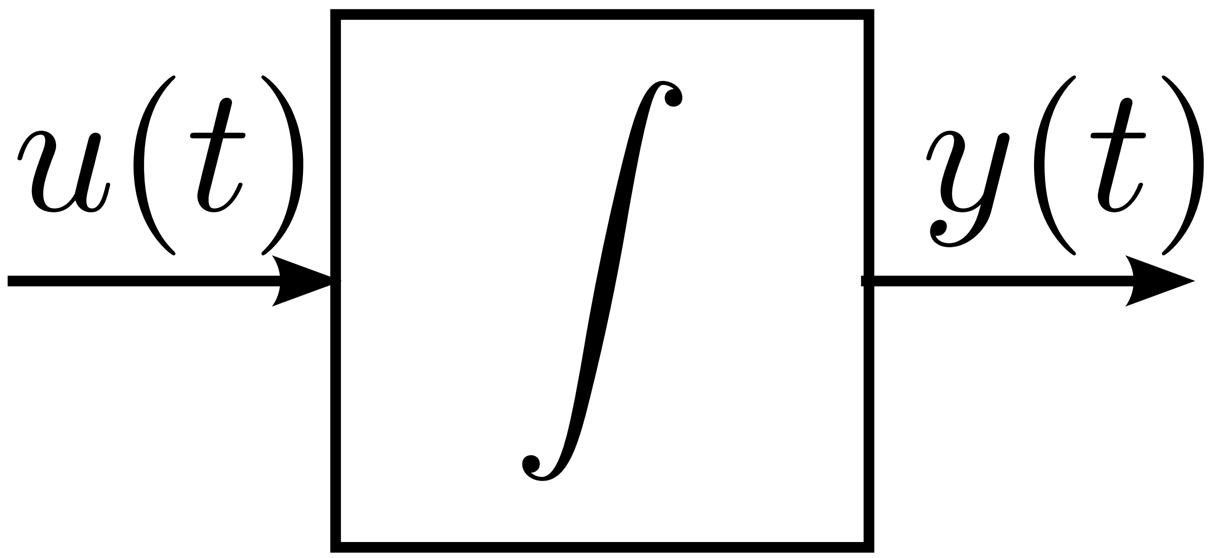


Toolkit

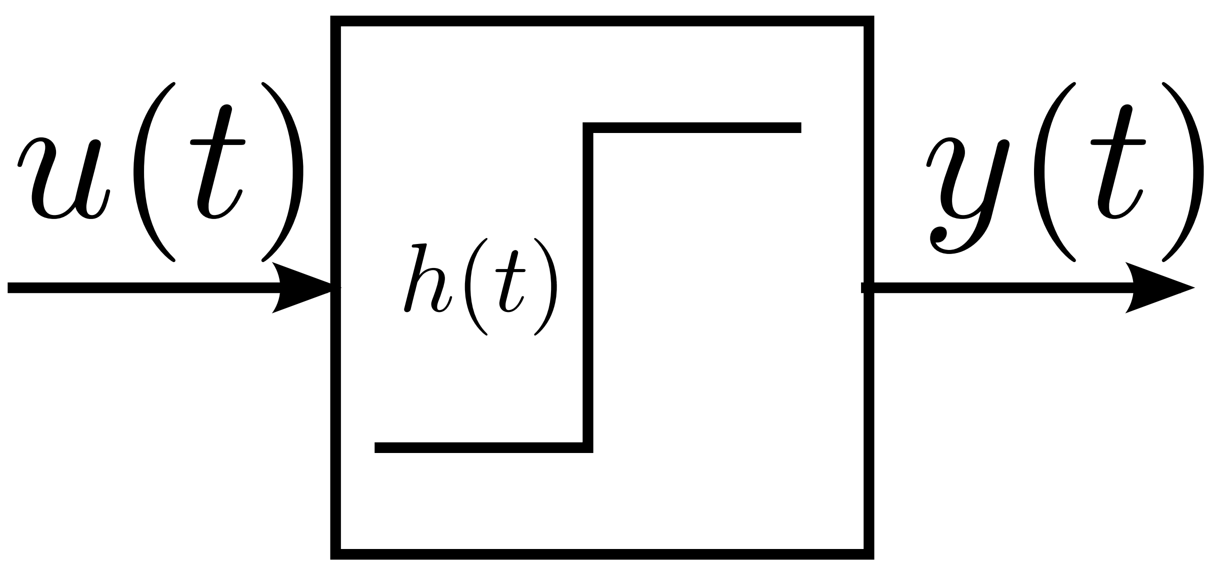


Block-Diagrams

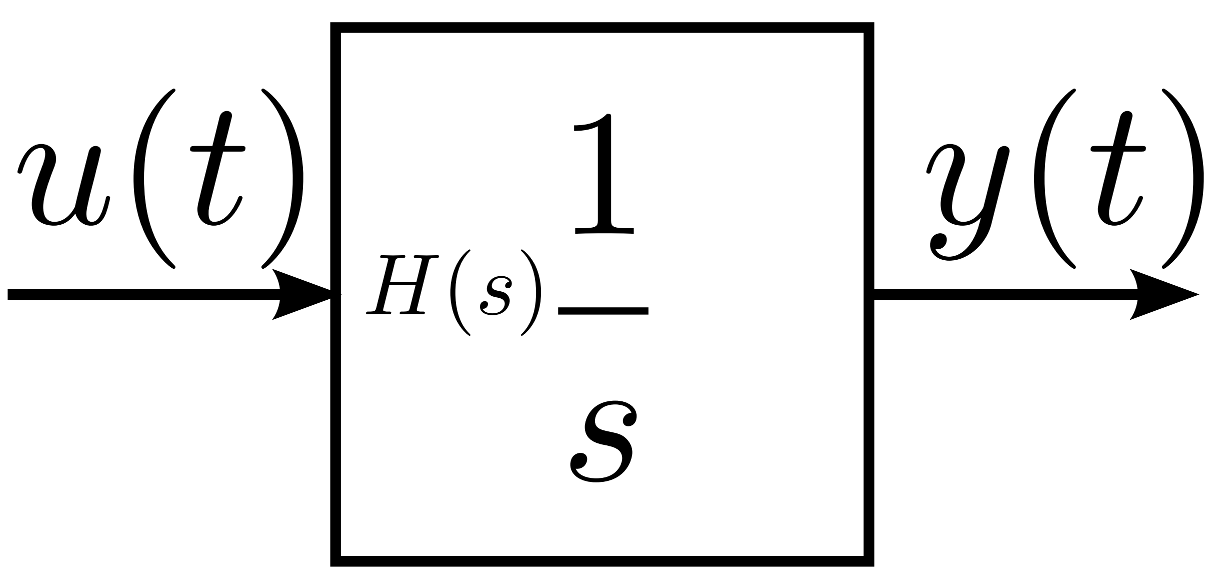
ODEs — Linear Time-Invariant



$$y(t) = y(0) + \int_0^t u(\theta)\, d\theta$$



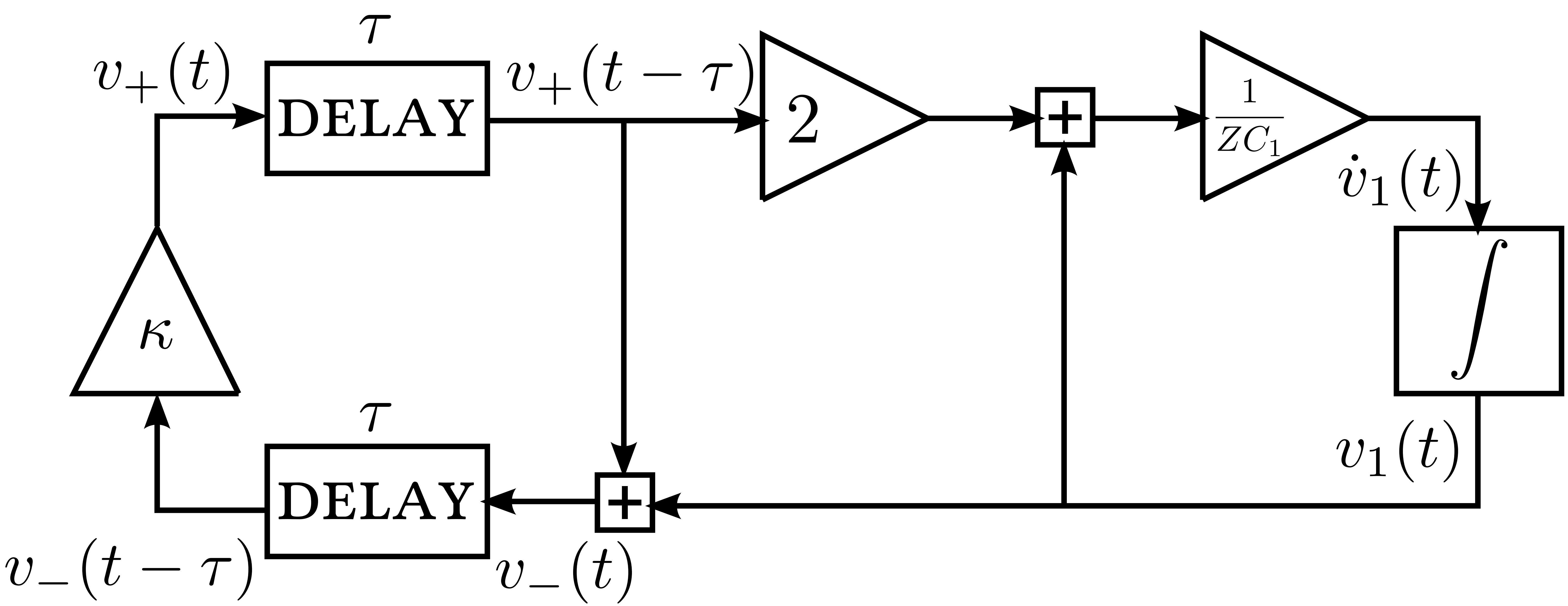
$$y(t) = y(0) + (h * u)(t)$$



$$H(s) = \mathscr{L}h(s)$$

Block-Diagrams

DDAEs — Linear Time-Invariant



Measures and Delays

$$\mathfrak{M}([0,\tau],\mathbb{R})$$

- Radon measure
- real-valued
- support $\subset [0,\tau]$

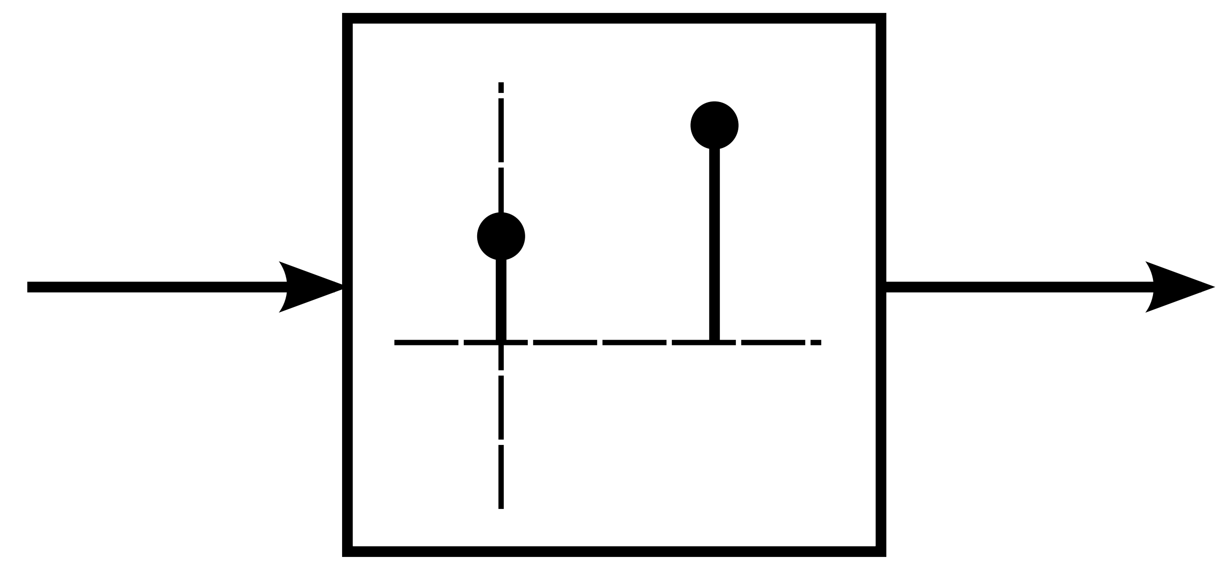
$$a \in \mathcal{L}(X^1,\mathbb{R}) \longleftrightarrow a^* \in \mathfrak{M}([0,\tau],\mathbb{R})$$

$$a\phi = \int_{[0,\tau]} x(-\theta)da^*(\theta)$$

Delays as Convolutions

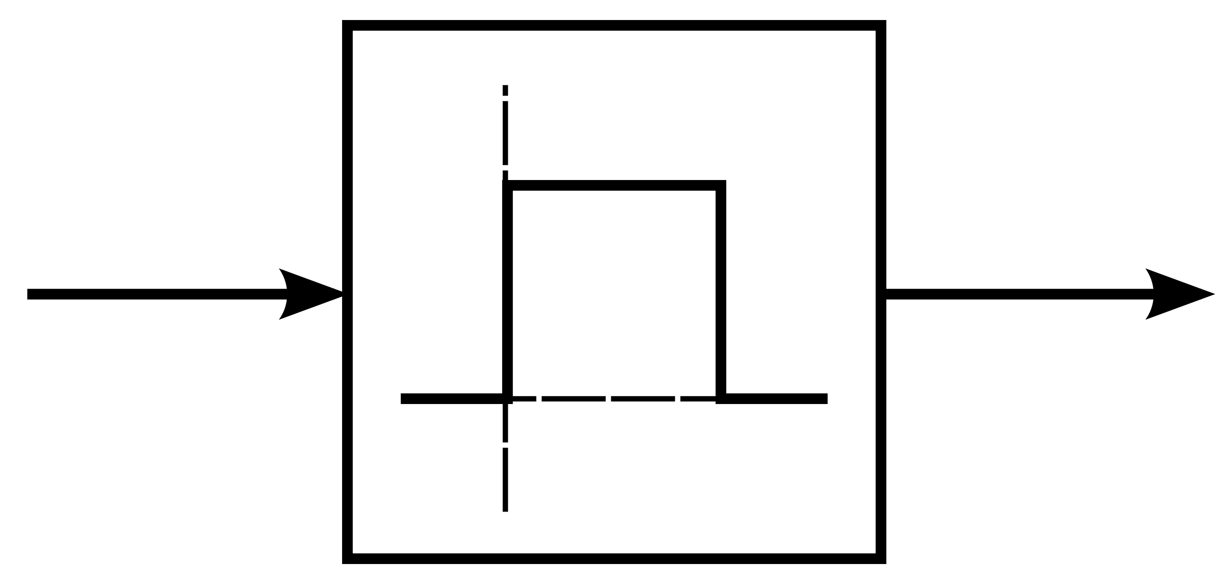
$$ax_t = \int_{[0,\tau]} x(t-\theta)da^*(\theta) = (a^* * x)(t)$$

$$a\phi = \phi(0)/2 + \phi(-1) \quad \longleftrightarrow$$



$$a^* = 1/2 \times \delta_0 + \delta_1$$

$$a\phi = \int_{[-1,0]} \phi(\theta)\, d\theta \quad \longleftrightarrow$$



$$a^* = \chi_{[0,1]}\lambda$$

Matrix-Valued Measures

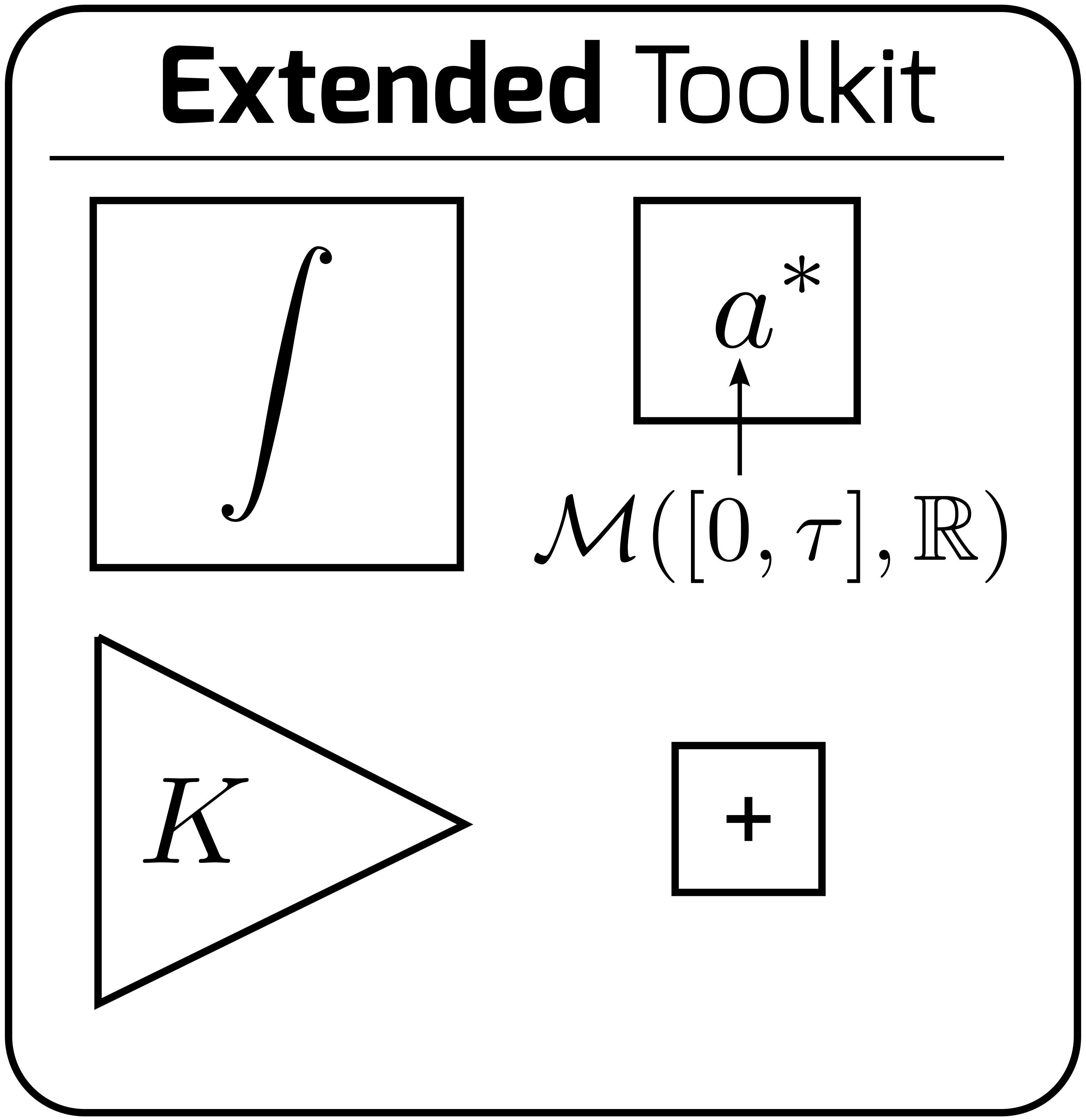
$$A \in \mathcal{L}(X^j, \mathbb{R}^i) \longleftrightarrow A^* \in \mathfrak{M}([0,\tau], \mathbb{R}^{i\times j})$$

$$Ax_t = \int_{[0,\tau]} dA^*(\theta) x(t-\theta) = (A^* * x)(t)$$

$$\downarrow$$

$$\sum_\ell \left[\int_{[0,\tau]} \sum_k x_k(t-\theta) dA^*_{\ell k}(\theta) \right] e_\ell$$

Block-Diagrams



combine →

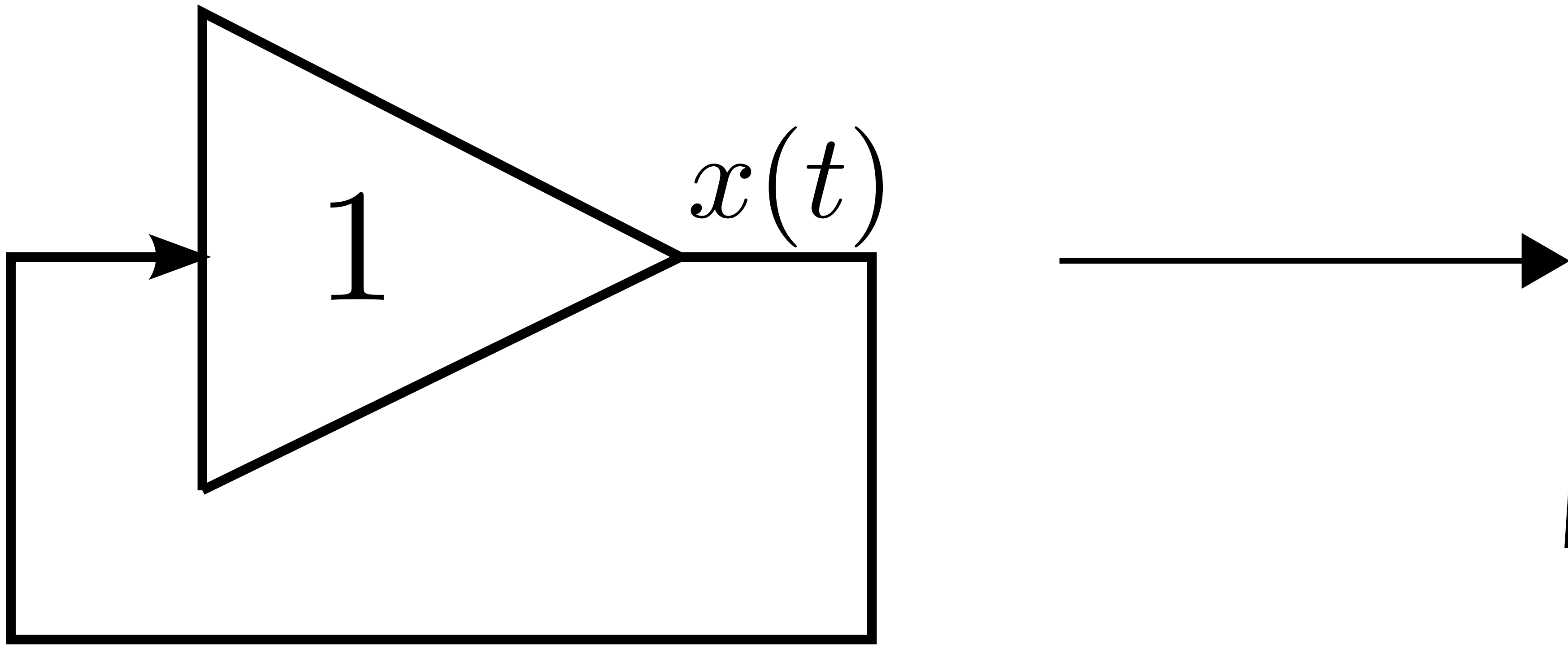
← decompose

DDAE

However ...

Existence/Uniqueness of the solutions are not guaranteed.

Algebraic/Causality Loops

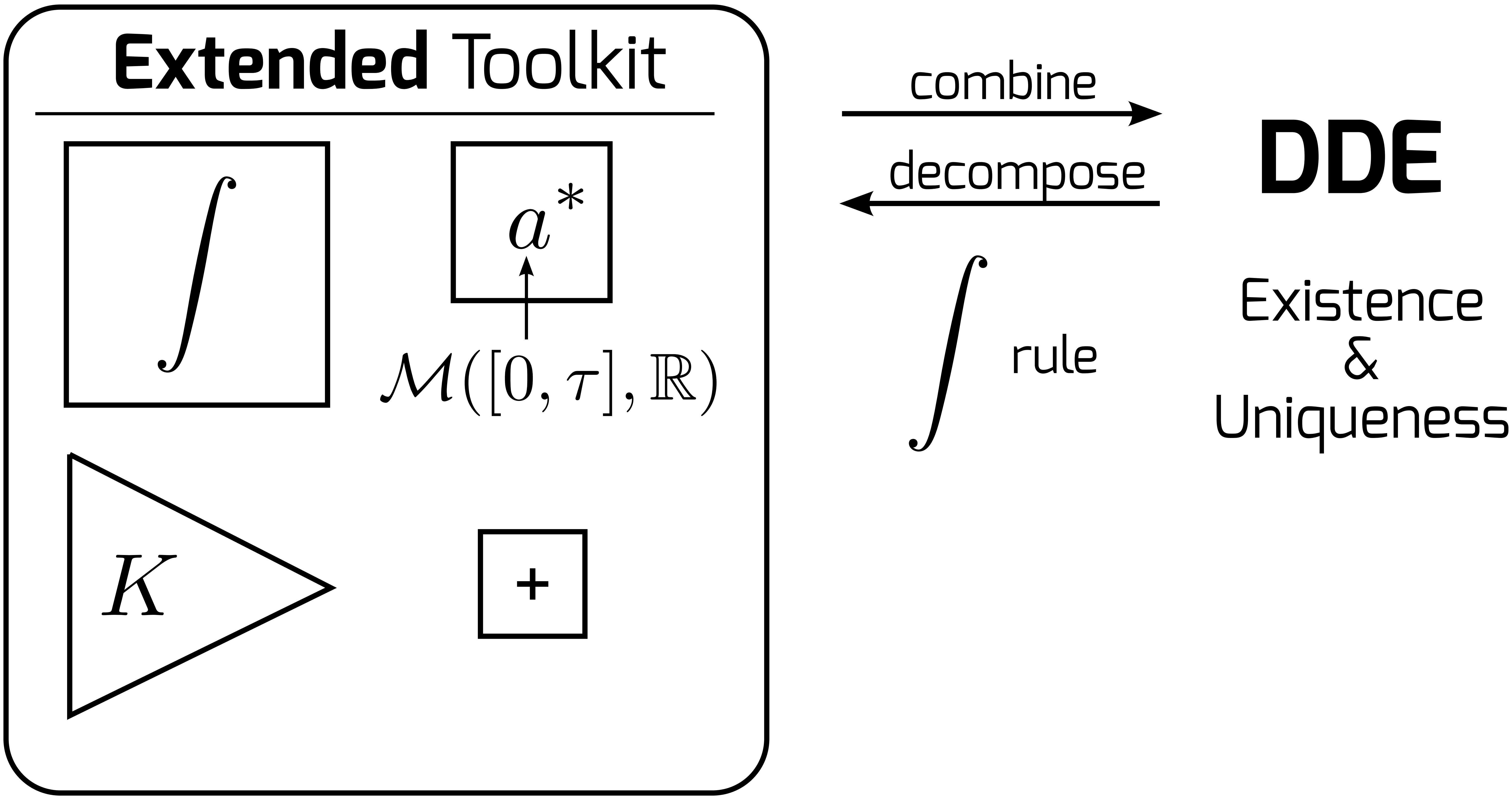


$$x(t) = x(t)$$

Multiple solutions

For ODEs,
one integrator in each diagram loop
ensures existence and uniqueness

Algebraic/Causality Loops



Causality Loops

The integrator rule should be rephrased:

“No loop without a strictly causal element.”

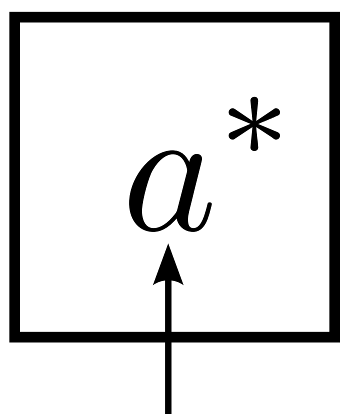
Laplace domain criterion:

$$H(s) \to 0 \text{ when } \Re s \to +\infty$$

Integrators are strictly causal: $H(s) = \dfrac{1}{s}$

Strict Causality

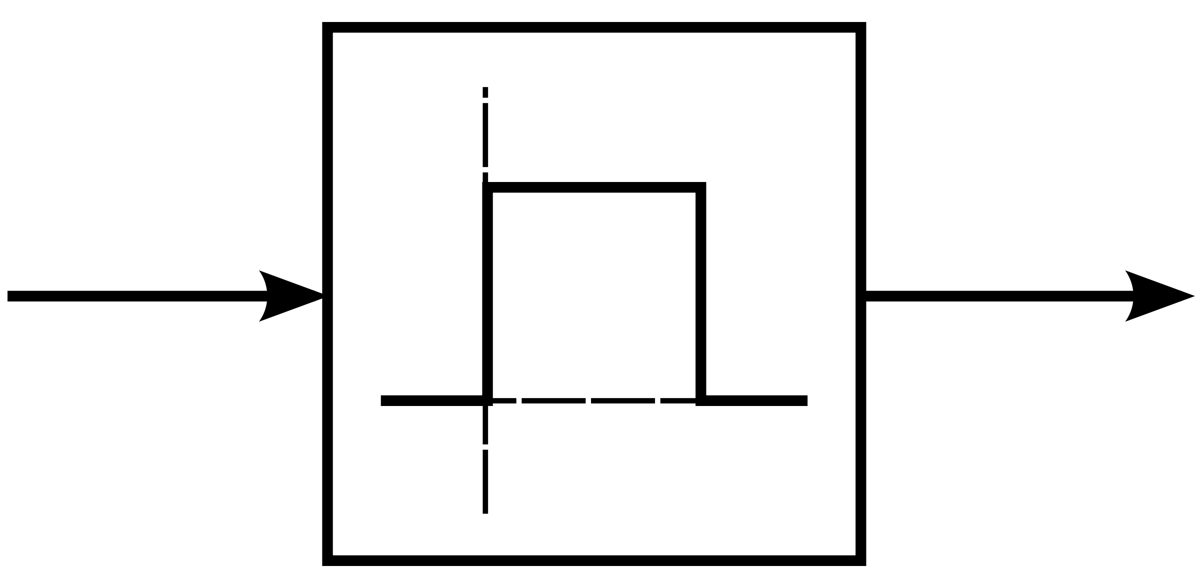
In the Time Domain



$\mathcal{M}([0, \tau], \mathbb{R})$

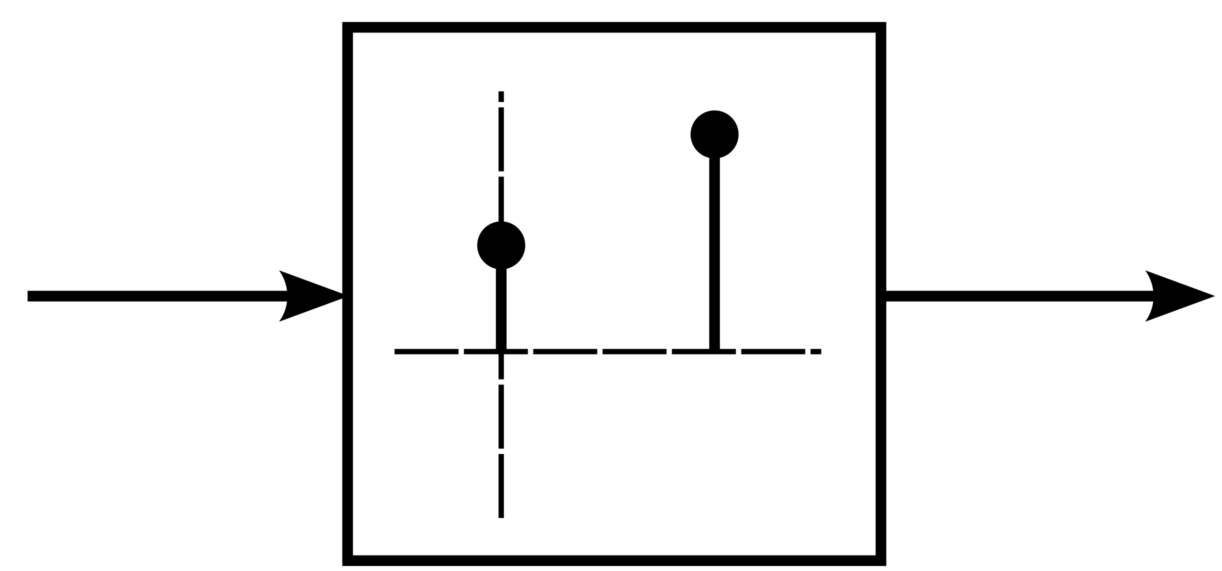
$$\lim_{\Re s \to +\infty} \mathscr{L} a^*(s) = a^*\{0\}$$

Strictly Causal



$a^* = \chi_{[0,1]}\lambda$

NOT Strictly Causal



$a^* = 1/2 \times \delta_0 + \delta_1$

Graph Theory

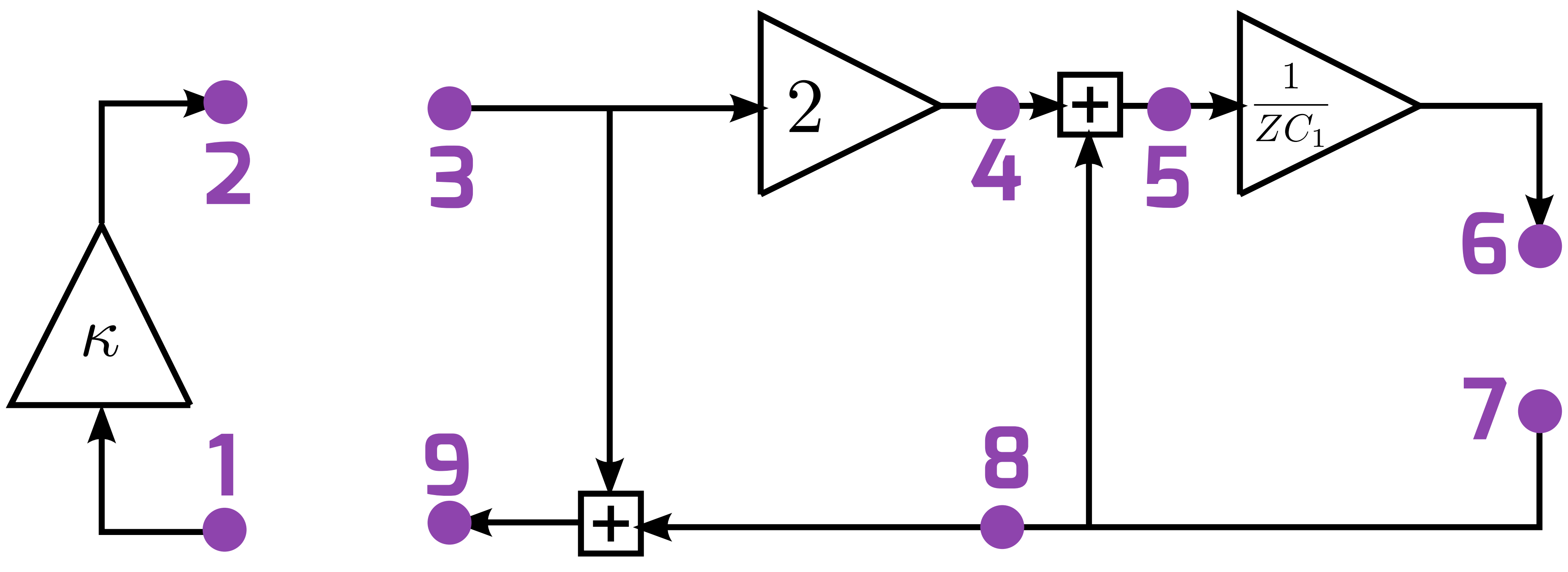
1 ▸ get rid of all strictly causal components.

2 ▸ number all nodes in the diagram.

3 ▸ define the **adjacency matrix:**

$$\mathcal{A}_{ij} = \text{number of edges } i \to j$$

Graph Theory



$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Graph Theory

No Algebraic Loop $\longleftrightarrow$ $\exists\, p,\ \mathcal{A}^p = 0$

$\updownarrow$

DDAE

$$\dot{x}(t) = Ax_t + By_t$$
$$y(t) = Cx_t + Dy_t$$

$$\exists\, p,\ [D^*\{0\}]^p = 0$$

$\downarrow$

Existence of $[I - D^*\{0\}]^{-1}$

Existence / Uniqueness

Product Space Approach

$$\begin{aligned} \dot{x}(t) &= Ax_t + By_t \\ y(t) &= Cx_t + Dy_t \end{aligned}$$

Existence of

$$[I - D^*\{0\}]^{-1}$$

Initial Values:

$$\begin{aligned} &x(0^+) \in \mathbb{R}^n \\ &x_0 \in L^1([-\tau, 0], \mathbb{R}^n) \\ &y_0 \in L^1([-\tau, 0], \mathbb{R}^m) \end{aligned}$$

Existence of Solutions

Convolution Equation

$$z : (0, +\infty) \to \mathbb{R}^{n+m}$$

$$z(t) = (x(t), y(t))$$

DDAE $\longrightarrow$

$$\left| \begin{array}{l} z = H * z + f \\ f = F(x(0^+), x_0, y_0) \end{array} \right.$$

Additionally $H\{0\} = 0$ (change of variables).

Existence of Solutions

Search for a solution z such that:

$$\|z^\sigma\|_1 = \int_0^{+\infty} |z(t)| \exp(-\sigma t) \, dt < +\infty$$

for σ large enough.

Properties

$$(H * z)^\sigma = H^\sigma * z^\sigma$$

$$\|H^\sigma\|_1 \to |H\{0\}| \quad \text{when} \quad \sigma \to +\infty$$

Search for Solutions

A solution z is a fixed point of:

$$\mathcal{F} : z^{\sigma} \to H^{\sigma} * z^{\sigma} + f^{\sigma}$$

As $H\{0\} = 0$, for large values of σ:

$$\|H^{\sigma}\|_1 < 1$$

and the mapping $\mathcal{F}$ is a contraction.

↓

Solution Existence and Uniqueness

DDAEs Well-Posedness

Assumption: $I - D\{0\}$ invertible.

There is a linear bounded mapping:

$$(x(0^+), x_0, y_0) \in \mathbb{R}^n \times L^2([-\tau, 0], \mathbb{R}^{m+n})$$

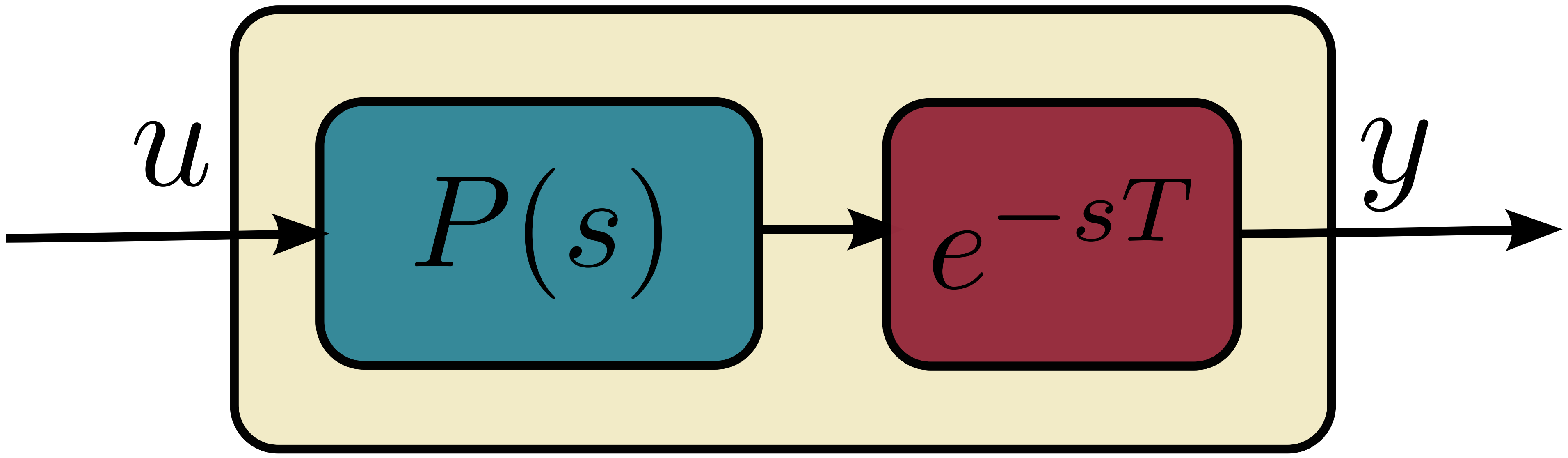
$$\downarrow$$

$$(x, y) \in W^{1,2}([0, T], \mathbb{R}^n) \times L^2([0, T], \mathbb{R}^m)$$

(Salamon, 1984).

Control & Stability
from the Smith predictor to Finite Spectrum Assignment

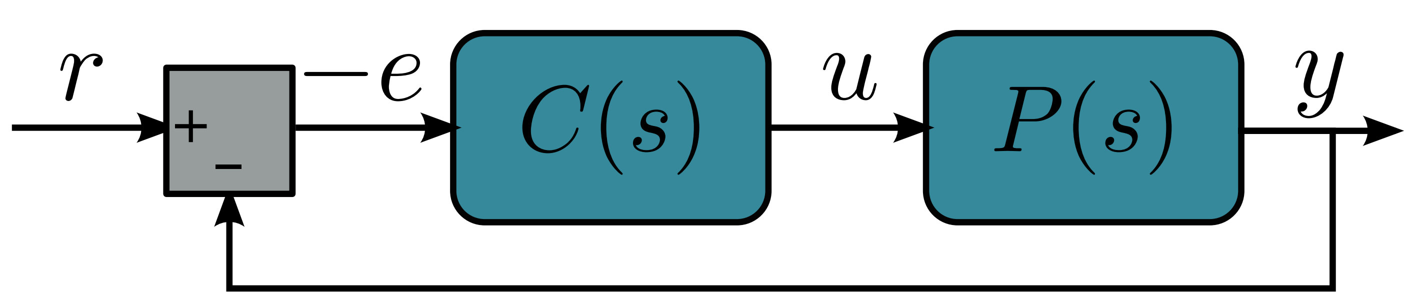
Dead-Time Systems



$$\left|\begin{array}{l}\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t - T)\end{array}\right.$$

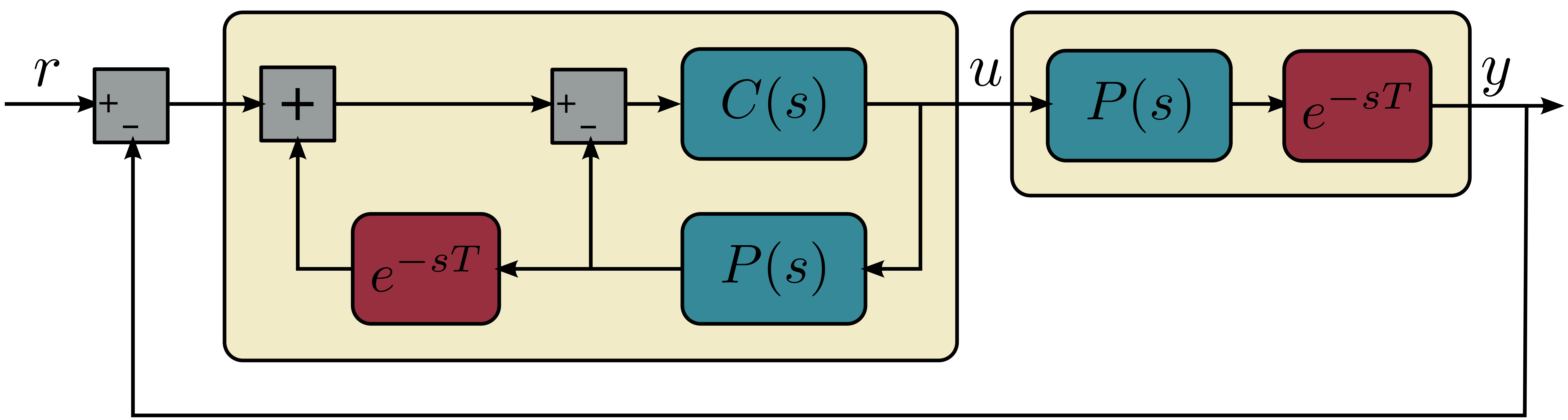
$$P(s) = C(sI - A)^{-1}B$$

Smith Predictor



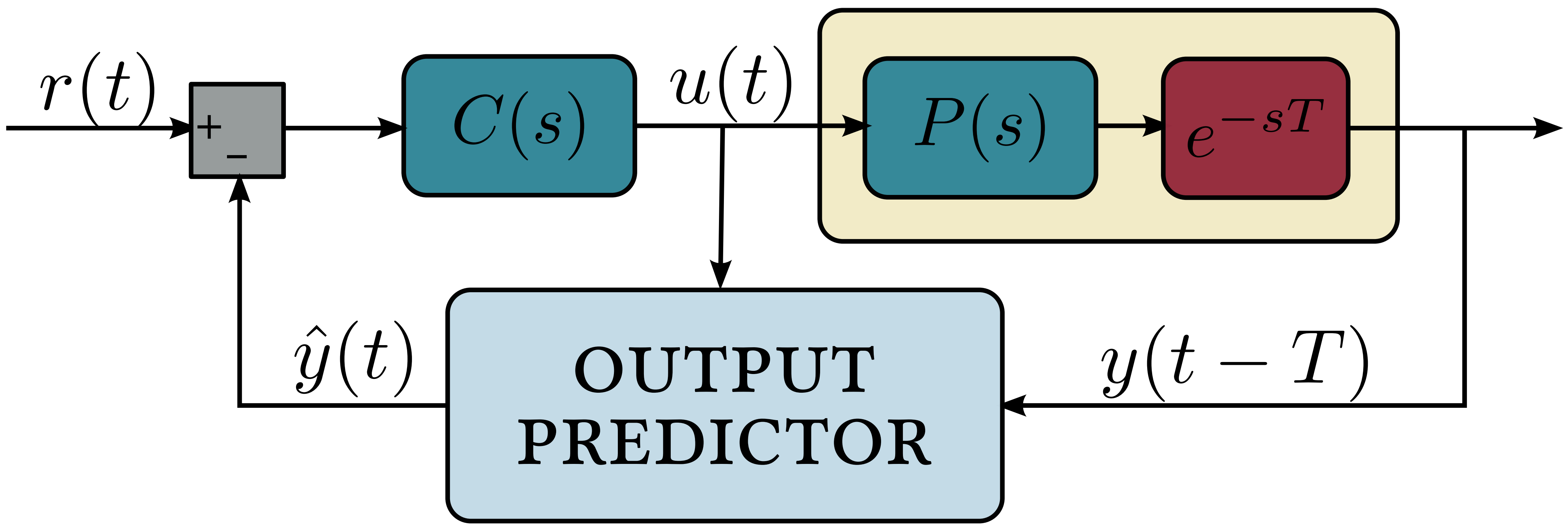
$$\dot{z} = Ez + F(-e)$$
$$u = Gz + H(-e)$$

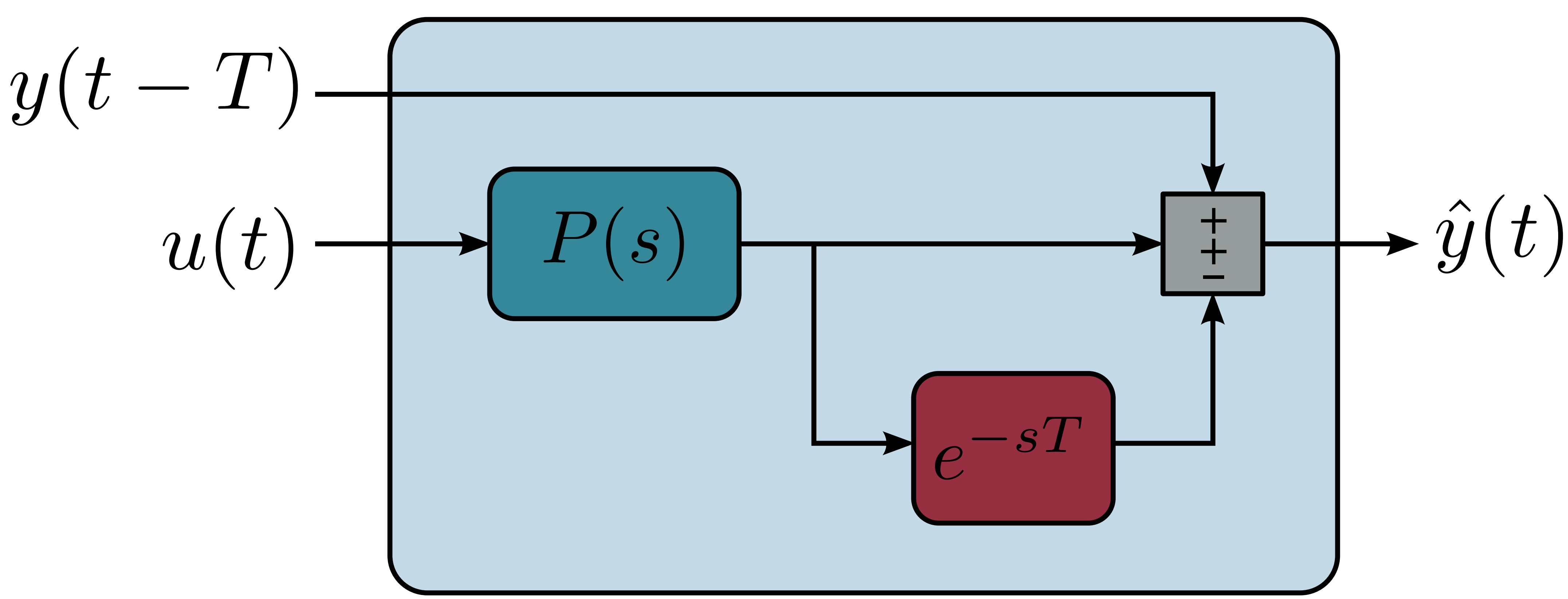
$$C(s) = G(sI - E)^{-1}F + H$$



Smith Predictor

Explained





State-Space Model

Delay-Free Dynamics

x : plant state
z : controller state

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix}(t) = M \begin{bmatrix} x \\ z \end{bmatrix}(t)$$

$$M = \begin{bmatrix} A - BHC & BG \\ -FC & E \end{bmatrix}$$

Exponential Stability:

$$\det\left(sI - M\right) = 0 \rightarrow \Re s < 0$$

State-Space Model

Delayed System + Smith Predictor

e : prediction error

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{e} \end{bmatrix}(t) = \mathcal{A}\begin{bmatrix} x \\ z \\ e \end{bmatrix}(t) + \mathcal{A}_d\begin{bmatrix} x \\ z \\ e \end{bmatrix}(t-T)$$

$$\mathcal{A} = \begin{bmatrix} A - BHC & BG & -BHC \\ -FC & E & -FC \\ 0 & 0 & A \end{bmatrix}$$

$$\mathcal{A}_d = \begin{bmatrix} 0 & 0 & BHC \\ 0 & 0 & FC \\ 0 & 0 & 0 \end{bmatrix}$$

Characteristic Matrix

Delay-Differential Equations

$$\dot{x}(t) = (A^* * x)(t)$$

Exponential time-dependent function

$$x(t) = x(0) \exp st$$

solution of the DDE iff:

$$\Delta(s)x(0) = 0$$

where:

$$\Delta(s) = [sI - \mathscr{L}(A^*)(s)]$$

Characteristic Equation

Delay-Differential Equations

Characteristic

Characteristic	matrix	$\Delta(s) = [sI - \mathscr{L}(A^*)(s)]$
	function	$s \in \mathbb{C} \to \det \Delta(s)$
	equation	$\det \Delta(s) = 0$

Roots of the charac. equation : system poles

Exponential Stability

Delay-Differential Equations

Spectrum Determined Growth:

$$\sup\{\Re s \mid s \in \mathbb{C},\ \det \Delta(s) = 0\} < 0$$

(e.g. Hale & al. 77/93,
Batkai & al. 05,
Bensoussan & al. 06)

Exponential Stability

Delay System + Smith Predictor

$$\Delta(s) = sI - \mathcal{A} - \mathcal{A}_d \exp(-sT)$$

$$\Delta(s) = \begin{bmatrix} sI - M & ? \\ 0 & sI - A \end{bmatrix}$$

closed-loop / delay-free poles

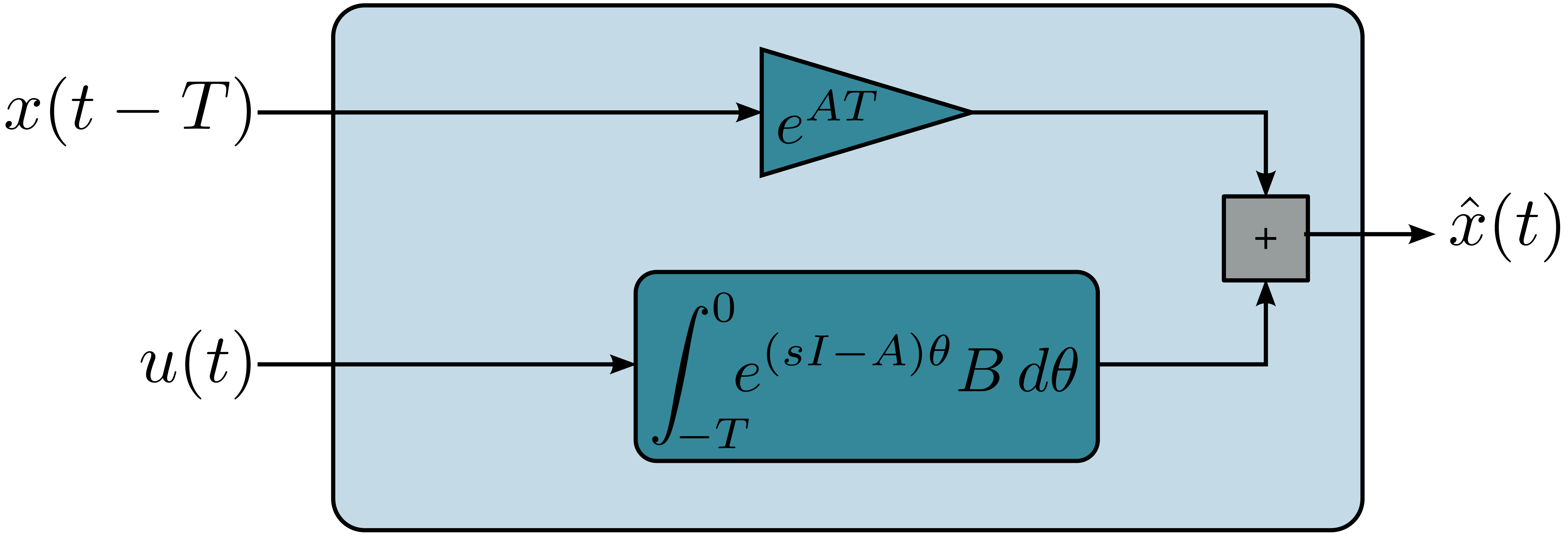
$$\det \Delta(s) = \det(sI - M) \times \det(sI - A)$$

open-loop poles

State Predictor

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\longrightarrow \quad x(t) = e^{AT}x(t-T) + \int_{[0,T]} [e^{A\theta}B]\, u(t-\theta)\, d\theta$$



State Predictor
Controller

Apply the control

$$u(t) = -K\hat{x}(t)$$

where K is selected such that

$$\left| \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ u(t) = -Kx(t) \end{array} \right.$$

is exponentially stable:

$$\det\left(sI - A + BK\right) = 0 \;\rightarrow\; \Re s < 0$$

State Predictor + Control

Closed-Loop Dynamics

$$\left| \begin{array}{l} \dot{x}(t) = Ax(t) - BK\hat{x}(t) \\ \hat{x}(t) = e^{AT}x(t-T) - \int_{[0,T]} \left[e^{A\theta}BKd\theta\right]\hat{x}(t-\theta) \end{array} \right.$$

DDAE with discrete + distributed delays

Exponential Stability

Delay-Differential Algebraic Equations

$$\Delta(s) = \begin{bmatrix} sI_n & 0 \\ 0 & I_m \end{bmatrix} - \mathscr{L}\begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}(s)$$

Spectrum Determined Growth:

$$\sup\{\Re s \mid s \in \mathbb{C},\ \det \Delta(s) = 0\} < 0$$

(Henry 74, Hale/Martinez-Amores 77,
Greiner/Schwarz 91, Hale/Verduyn Lunel 93,
..., Boisgérault 13)

State Predictor / FSA

Finite-Spectrum Assignment

$$\Delta(s) = \left[\begin{array}{c|c} sI - A & BK \\ \hline -e^{-(sI-A)T} & I + [sI - A]^{-1}(I - e^{-(sI-A)T})BK \end{array}\right]$$

$$\det \Delta(s) = \det(sI - A + BK)$$

closed-loop / delay free poles