The Winding Number

Sébastien Boisgérault, Mines ParisTech, under CC BY-NC-SA 4.0

September 30, 2019

Contents

Definitions 1
Properties 4
Simply Connected Sets 5
A Complex Analytic Approach 7
References 8

Definitions

The argument of a non-zero complex number is only defined modulo $2\pi$. A convenient way to describe mathematically this relationship is to associate to any such number the set of admissible values of its argument:

Definition – The Argument Function. The set-valued (or multi-valued) function $\text{Arg}$, defined on $\mathbb{C}^*$ by

$$\text{Arg} \, z = \left\{ \theta \in \mathbb{R} \mid e^{i\theta} = \frac{z}{|z|} \right\},$$

is called the argument function.

If we need a classic single-valued function instead, we have for example:

Definition – Principal Value of the Argument. The principal value of the argument is the unique continuous function

$$\text{arg} : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{R}$$

such that

$$\text{arg} \, 1 = 0$$
which is a choice of the argument on its domain:
\[ \forall z \in \mathbb{C} \setminus \mathbb{R}_-, \ \ arg z \in \text{Arg} z. \]

**Proof (existence and uniqueness).** Define arg on \( \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{R} \) by:
\[
\text{arg}(x + iy) = \begin{cases} 
\arctan y/x & \text{if } x > 0, \\
+\pi/2 - \arctan x/y & \text{if } y > 0, \\
-\pi/2 - \arctan x/y & \text{if } y < 0.
\end{cases}
\]

This definition is non-ambiguous: if \( x > 0 \) and \( y > 0 \), we have
\[ \arctan x/y + \arctan y/x = \pi/2 \]
and a similar equality holds when \( x > 0 \) and \( y < 0 \). As each of the three expressions used to define arg has an open domain and is continuous, the function itself is continuous. It is a choice of the argument thanks to the definition of \( \arctan \): for example, if \( x > 0 \), with \( \theta = \text{arg}(x + iy) \), we have
\[ \frac{\sin \theta}{\cos \theta} = \tan \theta = \tan(\arctan y/x) = \frac{y}{x}, \]
hence, as \( \cos \theta > 0 \) and \( x > 0 \), there is a \( \lambda > 0 \) such that
\[ x + iy = \lambda(\cos \theta + i \sin \theta) = \lambda e^{i \theta}, \]
This equation yields \( \text{arg} x + iy \in \text{Arg} x + iy \). The proof for the half-planes \( y > 0 \) and \( y < 0 \) is similar.

If \( f \) is another continuous choice of the argument on \( \mathbb{C} \setminus \mathbb{R}_- \) such that \( f(1) = 0 \), the image of \( \mathbb{C} \setminus \mathbb{R}_- \) by the difference \( f - \text{arg} \) is a subset of \( 2\pi \mathbb{Z} \) that contains 0, and it’s also path-connected as the image of a path-connected set by a continuous function. Consequently, it is the singleton \{0\}: \( f \) and \( \text{arg} \) are equal. \( \blacksquare \)

We cannot avoid the introduction of a cut in the complex plane when we search for a continuous choice of the argument: there is no continuous choice of the argument on \( \mathbb{C}^* \). However, for a continuous choice of the argument along a path of \( \mathbb{C}^* \), there is no such restriction:

The following theorem is a special case of the path lifting property (in the context of covering spaces; refer to (Hatcher 2002) for details).

**Theorem – Continuous Choice of the Argument.** Let \( a \in \mathbb{C} \) and \( \gamma \) be a path of \( \mathbb{C} \setminus \{a\} \). Let \( \theta_0 \in \mathbb{R} \) be a value of the argument of \( \gamma(0) - a \):
\[ \theta_0 \in \text{Arg}(\gamma(0) - a). \]

There is a unique continuous function \( \theta : [0, 1] \to \mathbb{R} \) such that \( \theta(0) = \theta_0 \) which is a choice of \( z \mapsto \text{Arg}(z - a) \) on \( \gamma \):
\[ \forall t \in [0, 1], \ \theta(t) \in \text{Arg}(\gamma(t) - a). \]
The uniqueness of a continuous choice is a consequence of the intermediate value theorem. With a new coordinate system to provide a global continuous choice of the argument.

**Definition – Path Exterior & Interior.** The *exterior* and *interior* of a closed path $\gamma$ are the subsets of the complex plane defined by

$\text{Ext} \, \gamma = \{ z \in \mathbb{C} \setminus \gamma([0, 1]) \mid \text{ind}(\gamma, z) = 0 \}.$

**Proof.** Let $(x(t), y(t))$ be the cartesian coordinates of $\gamma(t)$ in the system with origin $a$ and basis $(e^{i\theta_0}, ie^{i\theta_0})$. As long as $x(t) > 0$, the function

$$t \mapsto \theta_0 + \arg(x(t) + iy(t))$$

is a continuous choice of the argument of $\gamma(t) - a$. Let $d$ be the distance between $a$ and $\gamma([0, 1])$ and let $n \in \mathbb{N}$ such that

$$|t - s| \leq 2^{-n} \Rightarrow |\gamma(t) - \gamma(s)| < d.$$  

The condition $x(t) > 0$ is ensured for any $t$ in $[0, 2^{-n}]$. This construction of a continuous choice may be iterated locally on every interval $[k2^{-n}, (k + 1)2^{-n}]$ with a new coordinate system to provide a global continuous choice of the argument on $[0, 1]$.

The uniqueness of a continuous choice is a consequence of the intermediate value theorem: if we assume that there are two such functions $\theta_1$ and $\theta_2$ with the same initial value $\theta_0$, as $\theta_1(0) - \theta_2(0) = 0$, if $\theta_1(t) - \theta_2(t) \neq 0$ for some $t \in [0, 1]$, then either $|\theta_1(t) - \theta_2(t)| < \pi$, or there is a $\tau \in [0, t]$ such that $\theta_1(\tau) - \theta_2(\tau) \neq 0$ and $|\theta_1(\tau) - \theta_2(\tau)| < \pi$. In any case, there is a contradiction since all values of the argument differ of a multiple of $2\pi$.

**Definition – Variation of the Argument.** Let $a \in \mathbb{C}$ and $\gamma$ be a path of $\mathbb{C} \setminus \{a\}$. The *variation* of $z \mapsto \arg(z - a)$ on $\gamma$ is defined as

$$\text{var}(z \mapsto \arg(z - a))|_{\gamma} = \theta(1) - \theta(0)$$

where $\theta$ is a continuous choice of $z \mapsto \arg(z - a)$ on $\gamma$.

**Proof (unambiguous definition).** If $\theta_1$ and $\theta_2$ are two continuous choices of $z \mapsto \arg(z - a)$ on $\gamma$, for any $t \in [0, 1]$, they differ of a multiple of $2\pi$. As the function $\theta_1 - \theta_2$ is continuous, by the intermediate value theorem, it is constant. Hence

$$(\theta_1 - \theta_2)(1) = (\theta_1 - \theta_2)(0),$$

and $\theta_1(1) - \theta_1(0) \neq \theta_2(1) - \theta_2(0)$.

**Definition – Winding Number / Index.** Let $a \in \mathbb{C}$ and $\gamma$ be a closed path of $\mathbb{C} \setminus \{a\}$. The *winding number* or *index* of $\gamma$ around $a$ is the integer

$$\text{ind}(\gamma, a) = \frac{1}{2\pi} [z \mapsto \arg(z - a)]|_{\gamma}.$$  

**Proof – The Winding Number is an Integer.** Let $\theta$ be a continuous choice function of $z \mapsto \arg(z - a)$ on $\gamma$; as the path $\gamma$ is closed, $\theta(0)$ and $\theta(1)$, which are values of the argument of $\gamma(0) - a = \gamma(1) - a$, are equal modulo $2\pi$, hence $(\theta(1) - \theta(0))/2\pi$ is an integer.

**Definition – Path Exterior & Interior.** The *exterior* and *interior* of a closed path $\gamma$ are the subsets of the complex plane defined by

$\text{Ext} \, \gamma = \{ z \in \mathbb{C} \setminus \gamma([0, 1]) \mid \text{ind}(\gamma, z) = 0 \}.$
and

\[ \text{Int } \gamma = \mathbb{C} \setminus (\gamma(0,1) \cup \text{Ext } \gamma) = \{ z \in \mathbb{C} \setminus \gamma(0,1) \mid \text{ind}(\gamma, z) \neq 0 \}. \]

**Properties**

**Theorem – The Winding Number is Locally Constant.** Let \( a \in \mathbb{C} \) and \( \gamma \) be a closed path of \( \mathbb{C} \setminus \{a\} \). There is a \( \epsilon > 0 \) such that, for any \( b \in \mathbb{C} \) and any closed path \( \beta \), if

\[ |b - a| < \epsilon \text{ and } (\forall t \in [0,1], \ |\beta(t) - \gamma(t)| < \epsilon) \]

then \( \beta \) is a path of \( \mathbb{C} \setminus \{b\} \) and

\[ \text{ind}(\gamma, a) = \text{ind}(\beta, b). \]

**Proof.** Let \( \epsilon = d(a, \gamma(0,1))/2 \). If \( |b - a| < \epsilon \) and for any \( t \in [0,1], \ |\gamma(t) - \beta(t)| < \epsilon \), then clearly \( b \in \mathbb{C} \setminus \beta([0,1]) \). Additionally, for any \( t \in [0,1] \) there are values \( \theta_1 \) of \( \text{Arg}(\gamma(t) - a) \) and \( \theta_2 \) of \( \text{Arg}(\beta(t) - b) \) such that \( |\theta_1 - \theta_2| < \pi/2 \). If we select some values \( \theta_{1,0} \) of \( \text{Arg}(\gamma(0) - a) \) and \( \theta_{2,0} \) of \( \text{Arg}(\beta(0) - b) \) such that \( |\theta_{1,0} - \theta_{2,0}| < \pi/2 \), then the corresponding continuous choices \( \theta_1 \) and \( \theta_2 \) satisfy \( |\theta_1(t) - \theta_2(t)| < \pi/2 \) for any \( t \in [0,1] \). Consequently

\[ |\text{ind}(\gamma, a) - \text{ind}(\beta, b)| = \left| \frac{\theta_1(1) - \theta_1(0) - \theta_2(1) + \theta_2(0)}{2\pi} \right| < \frac{1}{2}. \]

As both winding numbers are integers, they are equal.

**Corollary – The Winding Number is Constant on Components.** Let \( \gamma \) be a closed path. The function

\[ z \in \mathbb{C} \setminus \gamma(0,1) \mapsto \text{ind}(\gamma, z) \]

is constant on each component of \( \mathbb{C} \setminus \gamma(0,1) \). If additionally the component is unbounded, the value of the winding number is zero.

**Proof.** The mapping \( z \mapsto \text{ind}(\gamma, z) \) is locally constant – and hence constant – on every connected component of \( \mathbb{C} \setminus \gamma(0,1) \). If \( a \) belongs to some unbounded component of this set, there is a \( b \) in the same component such that \( |b| = r = \max_{t \in [0,1]} |\gamma(t)| \). It is possible to connect \( b \) to any point \( c \) such that \( |c| = r \) by a circular path in \( \mathbb{C} \setminus \gamma(0,1) \), thus we may assume that \( b \in \mathbb{R}_- \). The function

\[ \theta : t \in [0,1] \mapsto \text{arg}(\gamma(t) - b) \]

\(^1\)Otherwise, by the intermediate value theorem, we could find some \( t \in [0,1] \) such that \( |\theta_1(t) - \theta_2(t)| = \pi/2 \), but then, for every value \( \theta_{1,1} \) of \( \text{Arg}(\gamma(t) - a) \) and \( \theta_{2,1} \) of \( \text{Arg}(\beta(t) - b) \), we would have

\[ \theta_{1,1} - \theta_{2,1} = \theta_1(t) - \theta_2(t) + 2\pi k \]

for some \( k \in \mathbb{Z} \). Therefore, the choice of \( \theta_{1,1} \) and \( \theta_{2,1} \) such that \( |\theta_{1,1} - \theta_{2,1}| < \pi/2 \) would be impossible.
is a continuous choice of $z \mapsto \text{Arg}(z - b)$ along $\gamma$ and it satisfies

$$\forall t \in [0, 1], \ |\theta(t)| = \left| \arctan \frac{\text{Im}(\gamma(t) - b)}{\text{Re}(\gamma(t) - b)} \right| < \arctan \frac{r}{|b| - r} < \frac{\pi}{2}.$$ 

As $\gamma$ is a closed path, $\theta(0)$ and $\theta(1)$ – which are equal modulo $2\pi$ – are actually equal and

$$\text{ind}(\gamma, a) = \text{ind}(\gamma, b) = \frac{\theta(1) - \theta(0)}{2\pi} = 0$$

as expected.

**Simply Connected Sets**

**Definition – Simply/Multiply Connected Set & Holes.** Let $\Omega$ be an open subset of the plane. A hole of $\Omega$ is a bounded component of its complement $\mathbb{C} \setminus \Omega$. The set $\Omega$ is simply connected if it has no hole (if every component of its complement is unbounded) and multiply connected otherwise.

**Examples.**

1. The open set $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x < -1 \text{ or } x > 1\}$ is not connected but it is simply connected: its complement has a unique component which is unbounded, hence it has no holes.

2. The open set $\Omega = \mathbb{C} \setminus \{2^{-n} \mid n \in \mathbb{N}\}$ is multiply connected: its holes are exactly the singletons of its complement.

Intuitively, we should be able to circle around any hole of $\Omega$ without leaving the set; this idea leads to an alternate characterization of simply connected sets.

**Theorem – Simply Connected Sets & The Winding Number.** An open subset $\Omega$ of the complex plane is simply connected if and only if the interior of any closed path $\gamma$ of $\Omega$ is included in $\Omega$:

$$\forall z \in \mathbb{C} \setminus \gamma([0, 1]), \ \text{ind}(\gamma, z) \neq 0 \Rightarrow z \in \Omega,$$

or equivalently, if the complement of $\Omega$ is included in the exterior of $\gamma$:

$$\forall z \in \mathbb{C} \setminus \Omega, \ \text{ind}(\gamma, z) = 0.$$

**Examples.**

1. If $\gamma$ is a closed path of $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x < -1 \text{ or } x > 1\}$ and $z \in \mathbb{C} \setminus \Omega$, since $\mathbb{C} \setminus \Omega$ is connected and unbounded, $z$ belongs to an unbounded component of $\mathbb{C} \setminus \gamma([0, 1])$. Thus $\text{ind}(\gamma, z) = 0$ for any $z \in \mathbb{C} \setminus \Omega$.

2. The open set $\Omega = \mathbb{C} \setminus \{2^{-n} \mid n \in \mathbb{N}\}$ is open and multiply connected: for example $\gamma = 1 + 1/4[0]$ is a path of $\Omega$, $z = 1$ is a point of $\mathbb{C} \setminus \Omega$ and $\text{ind}(\gamma, 1) = 1$. 

5
The collection of squares that intersect we also encircle an infinity of extra holes.

**Proof – Simply Connected Sets & The Winding Number.**

Assume that 

\[ \sum \]

\[ \text{ind}(\gamma, a) = 1. \]

Now if the line segment \( \gamma \) belongs to \( \Gamma \) and \( \gamma([0, 1]) \cap L \neq \varnothing \), then \( \gamma_{\leftarrow} \) also belongs to \( \Gamma' \); if we remove all such pairs from \( \Gamma \), the resulting collection \( \Gamma' \) also satisfies

\[ \sum_{\gamma \in \Gamma'} \frac{1}{2\pi} [z \mapsto \text{Arg}(z - a)]_{\gamma} = 1. \]
and by construction the image of any \( \gamma \) in \( \Gamma' \) is included in \( \Omega \). The original collection \( \Gamma \) is balanced: for any square vertex \( n \), the number of line segments with \( n \) as an initial point and with \( n \) as a terminal point is the same. The collection \( \Gamma' \) has the same property. Consequently, the line segments of \( \Gamma' \) may be assembled in a finite sequence of closed paths \( \gamma_1, \ldots, \gamma_n \) and

\[
\sum_{k=1}^{n} \text{ind}(\gamma_k, a) = 1.
\]

Every point of \( L \) is either an interior point of some square of the collection, or the limit of such point; anyway, that means that

\[
\forall z \in L, \sum_{k=1}^{n} \text{ind}(\gamma_k, z) = 1
\]

and thus that there is at least one path \( \gamma_k \) such that \( \text{ind}(\gamma_k, z) \neq 0 \).

**A Complex Analytic Approach**

If a closed path is rectifiable, we may compute its winding number as a line integral; to prove this, we need the:

**Lemma.** Let \( a \in \mathbb{C} \) and \( \gamma \) be a rectifiable path of \( \mathbb{C} \setminus \{a\} \). For any \( t \in [0, 1] \), let \( \gamma_t \) be the path such that for any \( s \in [0, 1] \), \( \gamma_t(s) = \gamma(ts) \). The function \( \mu : [0, 1] \rightarrow \mathbb{C} \), defined by

\[
\mu(t) = \int_{\gamma_t} \frac{dz}{z-a}
\]

satisfies

\[
\exists \lambda \in \mathbb{C}^*, \forall t \in [0, 1], e^{\mu(t)} = \lambda \times (\gamma(t) - a).
\]

**Proof.** We only prove the lemma under the assumption that \( \gamma \) is continuously differentiable; the rectifiable case is a straightforward extension.

We have for any \( t \in [0, 1] \)

\[
\mu(t) = \int_{\gamma_t} \frac{dz}{z-a} = \int_{0}^{1} \frac{\gamma'(ts) \times t}{\gamma(ts) - a} ds = \int_{0}^{t} \frac{\gamma'(s)}{\gamma(s) - a} ds,
\]

hence

\[
\mu'(t) = \frac{\gamma'(t)}{\gamma(t) - a}
\]

and the derivative of the quotient \( \phi(t) = e^{\mu(t)}/(\gamma(t) - a) \) satisfies

\[
\phi'(t) = \mu'(t)\phi(t) - \frac{\gamma'(t)}{\gamma(t) - a} \phi(t) = 0
\]
which yields the result. ■

**Theorem – The Winding Number as a Line Integral.** Let $a \in \mathbb{C}$ and $\gamma$ be a rectifiable path of $\mathbb{C} \setminus \{a\}$. Then

\[
[z \mapsto \text{Arg}(z - a)]_\gamma = \text{Im} \left( \int_\gamma \frac{dz}{z - a} \right).
\]

If the path $\gamma$ is closed, then

\[
\text{ind}(\gamma, a) = \frac{1}{i2\pi} \int_\gamma \frac{dz}{z - a}.
\]

**Proof.** We use the function $\mu$ of the previous lemma. Applying the modulus to both sides of the equation $e^{\mu(t)} = \lambda \times (\gamma(t) - a)$ provides $e^{\text{Re}(\mu(t))} = |\lambda| \times |\gamma(t) - a|$, hence

\[
e^{i\text{Im}(\mu(t))} = \frac{\lambda}{|\lambda|} \frac{\gamma(t) - a}{|\gamma(t) - a|}.
\]

The function $t \in [0,1] \mapsto \text{Im}(\mu(t))$ is – up to a constant – a continuous choice of $z \mapsto \text{Arg}(z - a)$ on $\gamma$. Consequently,

\[
[z \mapsto \text{Arg}(z - a)]_\gamma = \text{Im}(\mu(1)) - \text{Im}(\mu(0)) = \text{Im}(\mu(1)),
\]

which is the desired result.

If additionally $\gamma$ is a closed path, the equations

\[
\gamma(0) = \gamma(1) \quad \text{and} \quad e^{\text{Re}(\mu(t))} = |\lambda| \times |\gamma(t) - a|
\]
yield $e^{\text{Re}(\mu(0))} = e^{\text{Re}(\mu(1))}$ and hence $\text{Re}(\mu(1)) = \text{Re}(\mu(0)) = 0$. Thus,

\[
\text{ind}(\gamma, a) = \frac{1}{2\pi} \text{Im}(\mu(1)) = \frac{1}{i2\pi} \mu(1),
\]

which concludes the proof. ■

**References**
