Integral Representations

Sébastien Boisgérault, Mines ParisTech, under CC BY-NC-SA 4.0

November 28, 2017

Contents

Exercises

Functions of Several Complex Variables

Question

Let $n \geq 2$, let Ω be an open subset of $\mathbb{C}^n$ and let $f : \Omega \mapsto \mathbb{C}$ a continuous function. Show that f is complex-differentiable in Ω if and only if for any $(z_1, \dots, z_n) \in \Omega$, the partial function

$$f_{k,z} : w \mapsto f(z_1, \dots, z_{k-1}, w, z_{k+1}, \dots, z_n)$$

is holomorphic.

Answer

We may define the embedding functions $e_{k,z} : \mathbb{C} \to \mathbb{C}^n$ by

$$e_{k,z}(w) = (z_1, \dots, z_{k-1}, w, z_{k+1}, \dots, z_n).$$

It is plain that the $e_{k,z}$ are continuous. A function $f_{k,z}$ is defined on the preimage of the open set Ω by $e_{k,z}$ which is therefore an open set.

Assume that f is complex-differentiable; it is continuous. Additionally, $f_{k,z} = f \circ e_{k,z}$; as the function $e_{k,z}$ is complex-linear, it is complex-differentiable and

$f_{k,z}$ is complex-differentiable (or holomorphic) as the composition of complex-differentiable functions.

Conversely, if f if continuous and every partial function $f_{k,z}$ is complex-differentiable, the function f itself is complex-differentiable as every partial derivative $z \in \Omega \mapsto (\partial f / \partial z_k)(z)$ is continuous – not merely as a function of its k-th variable which is plain, but as a function of all its variables.

Let $z = (z_1, \ldots, z_n) \in \Omega$, let $c \in \mathbb{C}$ and $r > 0$ such that

$$\forall\, w \in \mathbb{C},\ |w - c| \leq r \ \rightarrow\ (z_1, \ldots, z_{k-1}, w, z_k, \ldots, z_n) \in \Omega.$$

Cauchy's formula, applied to the partial function $f_{k,z}$ for the path $\gamma = c + r[\circlearrowleft]$, provides

$$f(z_1, \ldots, z_n) = \frac{1}{i2\pi} \int_\gamma \frac{f(z_1, \ldots, z_{k-1}, w, z_{k+1}, \ldots, z_n)}{w - z_k} dw$$

The integrand is continuous with respect to the pair (z_1, w_1) and complex-differentiable with respect to z_1, thus we may compute the partial derivative of f with respect to z_k satisfies by differentiation under the integral sign:

$$\frac{\partial f}{\partial z_k}(z_1, \ldots, z_n) = \frac{1}{i2\pi} \int_\gamma \frac{f(z_1, \ldots, z_{k-1}, w, z_{k+1}, \ldots, z_n)}{(w - z_k)^2} dw.$$

As the function f is continuous, the partial derivative is also continous.