C1223 & S1224 – Complex Analysis
Final Examination

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Problem S

A function $F : \Omega \mapsto \mathbb{C}^{m \times n}$ where Ω is an open subset of $\mathbb{C}$ and m, n are positive integers is a *matrix-valued* function of a complex variable. It is characterized by the collection of its element functions $f_{kl} : \Omega \mapsto \mathbb{C}$:

$$\forall \, z \in \Omega, \; F(z) = \left[\begin{array}{cccc} f_{11}(z) & f_{12}(z) & \cdots & f_{1n}(z) \\ \vdots & \vdots & \dots & \vdots \\ f_{m1}(z) & f_{12}(z) & \cdots & f_{mn}(z) \end{array} \right]$$

Using element functions, we can readily adapt the tools of Complex Analysis:

- F is holomorphic if all its element functions are holomorphic; its derivative F' is defined element-wise:

$$\forall \, z \in \Omega, \; F'(z) = \left[\begin{array}{cccc} f'_{11}(z) & f'_{12}(z) & \cdots & f'_{1n}(z) \\ \vdots & \vdots & \dots & \vdots \\ f'_{m1}(z) & f'_{12}(z) & \cdots & f'_{mn}(z) \end{array} \right]$$

- the integral of F along some path (or path sequence) γ is defined – if and only if it is defined for all its element functions – by

$$\forall \, z \in \Omega, \; \int_\gamma F(z) \, dz = \left[\begin{array}{cccc} \int_\gamma f_{11}(z) \, dz & \int_\gamma f_{12}(z) \, dz & \cdots & \int_\gamma f_{1n}(z) \, dz \\ \vdots & \vdots & \dots & \vdots \\ \int_\gamma f_{m1}(z) \, dz & \int_\gamma f_{12}(z) \, dz & \cdots & \int_\gamma f_{mn}(z) \, dz \end{array} \right].$$

Linear algebra reminder. The spectrum of a square matrix A is the set $\sigma(A) = \{z \in \mathbb{C} \mid \det(zI - A) = 0\}$. The inverse matrix A^{-1} of an invertible matrix A depends continuously on the elements of A (as long as A stays invertible). Polynomials of a square matrix A (e.g. $I + 2A + A^2$) commute with each other; rational functions of A (e.g. $(I + A)(I - A)^{-1}$) also commute with each other whenever they are defined.

Questions

0. Prove that for any holomorphic functions $F : \Omega \to \mathbb{C}^{m\times n}$ and $G : \Omega \to \mathbb{C}^{n\times p}$, the matrix function product

$$FG : z \in \Omega \mapsto F(z)G(z) \in \mathbb{C}^{m\times p}$$

   is holomorphic and $(FG)' = F'G + FG'$.

Let $p \in \mathbb{N}^*$ and $A \in \mathbb{C}^{p\times p}$. We denote Ω the set of complex numbers z such that the matrix $I - zA$ is invertible and F the matrix-valued function defined by

$$F : z \in \Omega \mapsto [I - zA]^{-1}.$$

1. Show that Ω is open. Relate the set $\mathbb{C} \setminus \Omega$ of singularities of F and the spectrum of A.

2. Show that the function F is holomorphic and compute its derivative.

   Hint: define the difference quotient

$$\epsilon(z,h) = \frac{F(z+h) - F(z)}{h}$$

   then compute $\epsilon(z,h)[I - zA][I - (z+h)A]$ (optionally, consider first the case with $p = 1$ and $A = [a]$ for some $a \in \mathbb{C}$ to guess the expression of the derivative in the general case).

3. Prove that the n-th order derivative $F^{(n)}$ of F exists for every natural number n and satisfies

$$\forall\, z \in \Omega,\ F^{(n)}(z) = n!A^n[I - zA]^{-n-1}.$$

   Hint: prove the result by induction on n.

4. What is the (possibly infinite) radius $r(A)$ of the largest open disk centered on the origin and included in Ω ? Relate $r(A)$ and the spectral radius of A, defined as

$$\rho(A) = \max\{|\lambda| \mid \lambda \in \sigma(A)\}.$$

   Find a sequence of matrices $A_n \in \mathbb{C}^{p\times p}$ such that

$$\forall\, z \in D(0, r(A)),\ F(z) = [I - zA]^{-1} = \sum_{n=0}^{+\infty} A_n z^n.$$

   (the convergence of this matrix series should be interpreted element-wise).

5. Let $n \in \mathbb{N}$. Let $z \in \mathbb{C} \setminus \sigma(A)$ such that $z \neq 0$; write $z^n[zI - A]^{-1}$ as a function of z and $F(z^{-1})$. Show that the function

$$z \in \mathbb{C} \setminus \sigma(A) \mapsto z^n[zI - A]^{-1}$$

   is complex-differentiable.

6. Compute the line integral

$$\frac{1}{i2\pi}\int_\gamma z^n[zI-A]^{-1}dz$$

first when $\gamma = r[\circlearrowleft]$ with $r > \rho(A)$, then more generally for any sequence of rectifiable closed paths γ of $\mathbb{C}\setminus\sigma(A)$ such that $\forall\,\lambda\in\sigma(A)$, $\mathrm{ind}(\gamma,\lambda)=1$.

7. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and let γ be a sequence of rectifiable closed paths of $\mathbb{C}\setminus\mathbb{R}_-$ such that $\forall\,\lambda\in\sigma(A)$, $\mathrm{ind}(\gamma,\lambda)=1$. Compute $[zI-A]^{-1}$ and then the matrix $\log A$, defined as

$$\log A = \frac{1}{i2\pi}\int_\gamma (\log z)[zI-A]^{-1}\,dz.$$

Answers

0. For any $z\in\Omega$, $j\in\{1,\dots,m\}$ and $l\in\{1,\dots,p\}$,

$$FG(z)_{jl} = \sum_{k=1}^n F(z)_{jk}G(z)_{kl} = \sum_{k=1}^n f_{jk}(z)g_{kl}(z).$$

By the sum and product rules for complex-valued holomorphic functions, every element of FG is holomorphic and

$$\begin{aligned}(FG)'(z)_{jl} &= \sum_{k=1}^n f'_{jk}(z)g_{kl}(z) + \sum_{k=1}^n f_{jk}(z)g'_{kl}(z)\\ &= F'G(z)_{jl} + FG'(z)_{jl}.\end{aligned}$$

1. The function $d : z\in\mathbb{C}\mapsto \det(I-zA)$ is a polynomial, thus it is continuous; the matrix $I-zA$ is invertible if and only if $d(z)\neq 0$, hence $\Omega = d^{-1}(\mathbb{C}^*)$ is open as the preimage of an open set by a continuous function.

   Since the matrix $I-zA$ is invertible for $z=0$, we have $0\in\Omega$; if $z\neq 0$,

$$I - zA = z(z^{-1}I - A)$$

   and it is invertible if and only if z^{-1} is not an eigenvalue of A. Finally,

$$\mathbb{C}\setminus\Omega = \{\lambda^{-1}\mid \lambda\in\sigma(A)\setminus\{0\}\}.$$

2. For any pair $(k,l)\in\{1,\dots,p\}^2$, any $z\in\Omega$ and $h\in\mathbb{C}$ small enough,

$$\epsilon(z,h)_{kl} = \frac{f_{kl}(z+h)-f_{kl}(z)}{h}$$

is the difference quotient of f_{kl} at z. Therefore, F is complex-differentiable at z if (and only if) $\epsilon(z, h)$ has a limit element-wise when $h \to 0$ and $F'(z)$ is this limit. Now, we have

$$\epsilon(z, h) = \frac{[I - (z+h)A]^{-1} - [I - zA]^{-1}}{h}$$

The matrices $I - (z+h)A$ and $I - zA$ commute, hence

$$\epsilon(z, h)[I - (z+h)A][I - zA] = \frac{[I - zA] - [I - (z+h)A]}{h} = A,$$

thus $\epsilon(z, h) = A[I - zA]^{-1}[I - (z+h)A]^{-1}$. Since matrix inversion is continuous

$$F'(z) = \lim_{h \to 0} \epsilon(z, h) = A[I - zA]^{-2}.$$

3. The function F is holomorphic; all of its elements are holomorphic, thus they have derivatives at all orders and so does F. Assume that the $n+1$-th order derivative of F satisfies

$$F^{(n+1)}(z) = nA[I - zA]^{-1}F^{(n)}(z).$$

This property certainly holds for $n = 0$, as $F'(z) = A[I - zA]^{-1}F(z)$. Now, by the product rule, as A and $F(z) = [I - zA]^{-1}$ commute,

$$\begin{aligned}
F^{(n+2)}(z) &= nAF'(z)F^{(n)}(z) + nAF(z)F^{(n+1)}(z) \\
&= AF(z)(nAF(z)F^{(n)}(z)) + nAF(z)F^{(n+1)}(z) \\
&= AF(z)F^{(n+1)}(z) + nAF(z)F^{(n+1)}(z) \\
&= (n+1)AF(z)F^{(n+1)}(z)
\end{aligned}$$

Consequently, the formula holds for all n, which leads to

$$F^{(n)}(z) = n!A^n[I - zA]^{-n-1}.$$

4. The radius $r(A)$ of the largest disk centered on the origin and included in Ω is

$$r(A) = \min\{|\lambda^{-1}| \mid \lambda \in \sigma(A) \setminus \{0\}\}$$

if $\sigma(A) \setminus \{0\}$ is non-empty and $r(A) = +\infty$ otherwise. If we adopt the convention $0^{-1} = +\infty$, we can rewrite this as

$$r(A) = (\max\{|\lambda| \mid \lambda \in \sigma(A)\})^{-1} = \rho(A)^{-1}.$$

The function F is holomorphic in Ω, thus every element f_{kl} is holomorphic in Ω, which has a Taylor series expansion in $D(0, r)$:

$$\forall\, z \in D(0, r),\ f_{kl}(z) = \sum_{n=0}^{+\infty} \frac{f_{kl}^{(n)}(0)}{n!} z^n.$$

Consequently,

$$\forall\, z \in D(0,r),\ F(z) = \sum_{n=0}^{+\infty} \frac{F^{(n)}(0)}{n!} z^n = \sum_{n=0}^{+\infty} A^n z^n.$$

5. If $z \in \mathbb{C} \setminus \sigma(A)$ and $z \neq 0$, then

$$z^n [zI - A]^{-1} = z^{n-1}[I - z^{-1}A]^{-1} = z^{n-1} F(z^{-1}).$$

Consequently, $z \in \mathbb{C} \setminus \sigma(A) \mapsto z^n[zI - A]^{-1}$ is complex-differentiable at z – by the product and chain rules – if $z \neq 0$, which is all we need unless $0 \not\in \sigma(A)$. In this case, A is invertible and by continuity of the matrix inversion, the function $z \mapsto z^n[zI - A]^{-1}$ is continuous at $z = 0$. Hence, 0 is a removable singularity of $z \mapsto z^{n-1}F(z^{-1})$ and the function $z \in \mathbb{C} \setminus \sigma(A) \mapsto z^n[zI - A]^{-1}$ is also complex-differentiable at $z = 0$.

6. If $|z| > \rho(A)$, $z \in \mathbb{C} \setminus \sigma(A)$, $|z^{-1}| < r(A)$ and

$$z^n[zI - A]^{-1} = z^{n-1}[I - z^{-1}A]^{-1} = z^{n-1} \sum_{l=0}^{+\infty} A^l z^{-l} = \sum_{l=0}^{+\infty} A^l z^{n-1-l}.$$

This convergence if uniform (element-wise) on compact subsets of the annulus, therefore

$$\begin{aligned} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} z^n [zI - A]^{-1} dz &= \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \sum_{l=0}^{+\infty} A^l z^{n-1-l} dz \\ &= \sum_{l=0}^{+\infty} A^l \left[\frac{1}{i2\pi} \int_{r[\circlearrowleft]} z^{n-1-l} dz \right] \end{aligned}$$

The integral $\int_{r[\circlearrowleft]} z^p \, dz$ is zero unless the integer p is -1, in which case it is equal to $i2\pi$, hence

$$\frac{1}{i2\pi} \int_{r[\circlearrowleft]} z^n [zI - A]^{-1} dz = A^n$$

Now, If γ is a sequence of rectifiable closed paths of $\mathbb{C} \setminus \sigma(A)$ such that $\forall\, \lambda \in \sigma(A)$, $\mathrm{ind}(\gamma, \lambda) = 1$ and $r > \rho(A)$, the sequence of paths $\mu = r[\circlearrowleft] \,|\, \gamma^{\leftarrow}$ satisfies $\forall\, z \in \sigma(A)$, $\mathrm{ind}(\mu, z) = 0$. Cauchy's integral theorem provides $\int_\mu z^n [zI - A]^{-1} \, dz = 0$, or equivalently,

$$\frac{1}{i2\pi} \int_\gamma z^n [zI - A]^{-1} \, dz = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} z^n [zI - A]^{-1} \, dz = A^n.$$

7. We have

$$zI - A = \left[\begin{array}{rr} z & +1 \\ -1 & z \end{array} \right]$$

hence $\det(zI - A) = z^2 + 1$. The spectrum of A is $\sigma(A) = \{-i, +i\}$ and

$$[zI - A]^{-1} = \frac{1}{z^2+1} \left[\begin{array}{rr} z & -1 \\ +1 & z \end{array} \right].$$

Consequently,

$$\frac{1}{i2\pi} \int_\gamma (\log z)[zI - A]^{-1}\, dz = \frac{1}{i2\pi} \int_\gamma \left[\begin{array}{rr} f(z) & -g(z) \\ g(z) & f(z) \end{array} \right] dz$$

with

$$f(z) = \frac{z \log z}{z^2+1},\ g(z) = \frac{\log z}{z^2+1}.$$

Let $\Omega = \mathbb{C} \setminus \mathbb{R}_- \setminus \sigma(A)$; this set is open. Both functions f and g are holomorphic on Ω, have isolated singularities in $\sigma(A)$ and are the quotient of holomorphic functions a and b defined on $\Omega \cup \sigma(A)$ such that when $\lambda \in \sigma(A)$, $a(\lambda) \neq 0$, $b(\lambda) = 0$ and $b'(\lambda) \neq 0$. Thus, we have

$$\sum_{z \in \sigma(A)} \operatorname{ind}(\gamma, z) \operatorname{res}(f, z) = \frac{(-i)\log(-i)}{2(-i)} + \frac{i \log i}{2i} = 0$$

and

$$\sum_{z \in \sigma(A)} \operatorname{ind}(\gamma, z) \operatorname{res}(g, z) = \frac{\log(-i)}{2(-i)} + \frac{\log i}{2i} = \frac{\pi}{2}.$$

Since the open set

$$\Omega \cup \sigma(A) = \mathbb{C} \setminus \mathbb{R}_-$$

is simply connected (it is star-shaped), $\operatorname{Int} \gamma \subset \Omega \cup \sigma(A)$. Finally, the residue theorem provides

$$\log A = \frac{1}{i2\pi} \int_\gamma (\log z)[zI - A]^{-1}\, dz = \frac{\pi}{2} \left[\begin{array}{rr} 0 & -1 \\ +1 & 0 \end{array} \right]$$

Problem L

A function $f : \mathbb{C} \to \mathbb{C}$ is *of exponential type* if

$$\exists\, \sigma \in \mathbb{R},\ \exists\, \kappa > 0, \forall\, z \in \mathbb{C},\ |f(z)| \leq \kappa e^{\sigma |z|}.$$

We define $\sigma(f)$ as the infimum of the exponential bounds σ:

$$\sigma(f) = \inf \{\sigma \in \mathbb{R} \mid \exists\, \kappa > 0, \forall\, z \in \mathbb{C},\ |f(z)| \leq \kappa e^{\sigma |z|}\}.$$

Questions

We suppose that f is complex-differentiable on $\mathbb{C}$ and denote

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

its Taylor expansion at 0.

1. Show that if f is of exponential type, $\sigma(f) \geq 0$ unless f is identically zero.
2. Show that the function
$$f_0 : z \in \mathbb{C} \to e^{-z}$$
is of exponential type and compute $\sigma(f_0)$.
3. Prove that if for some $\kappa > 0$ and $\sigma \geq 0$, $\forall\, n \in \mathbb{N}$, $|n! a_n| \leq \kappa \sigma^n$, then f is of exponential type and $\sigma(f) \leq \sigma$.
4. Prove that if $\forall\, z \in \mathbb{C},\ |f(z)| \leq \kappa e^{\sigma |z|}$ then
$$\forall\, n \in \mathbb{N},\ \forall\, r > 0,\ |a_n| \leq \kappa \frac{e^{\sigma r}}{r^n}.$$
Let $n \in \mathbb{N}$; what is the value of r that provides the tightest upper bound on $|a_n|$? Compute this bound.
5. Prove that if f is of exponential type and $\sigma > \sigma(f)$, there is a $\kappa > 0$ such that $\forall\, n \in \mathbb{N}$, $|n! a_n| \leq \kappa \sigma^n$. Reminder – Stirling's formula:
$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \text{ that is } \lim_{n \to +\infty} \frac{n! e^n}{\sqrt{2\pi n} n^n} = 1.$$
6. Suppose that f is of exponential type; show that the function $g : \mathbb{C} \to \mathbb{C}$ defined by
$$g(z) = \sum_{n=0}^{+\infty} |a_n| z^n$$
is of exponential type and $\sigma(g) = \sigma(f)$.
7. What is the domain of the Laplace transform of the function f_0? Compute $\mathcal{L}[f_0](s)$ for any s in this domain. Prove that in general the domain of the Laplace transform of a function of exponential type f contains the set $\{s \in \mathbb{C} \mid \text{Re } s > \sigma(f)\}$.
8. Show that whenever $\text{Re } s > \sigma(f)$,
$$\mathcal{L}[f](s) = \int_{\mathbb{R}_+} f(t) e^{-st}\, dt = \sum_{n=0}^{+\infty} \frac{n! a_n}{s^{n+1}}.$$

Prove that $\mathcal{L}[f]$ has a unique holomorphic extension to the open set

$$\Omega = \operatorname{dom} \mathcal{L}[f] \cup \{s \in \mathbb{C} \mid |s| > \sigma(f)\}.$$

We denote $\overline{\mathcal{L}}[f]$ this extension. Compute $\overline{\mathcal{L}}[f_0]$.

9. Compute for any $t \geq 0$ and any $r > \sigma(f)$ the line integral

$$\frac{1}{i2\pi} \int_{r[\circlearrowleft]} \overline{\mathcal{L}}[f] e^{st} \, ds.$$

Answers

1. If $\sigma(f) < 0$ then $\sigma = 0$ is an exponential bound of f: there is a $\kappa > 0$ such that $\forall\, z \in \mathbb{C}$, $|f(z)| \leq \kappa$. The function f is entire and bounded, thus by Liouville's theorem, it is identically zero.

2. For any complex number z,

$$|f_0(z)| = |e^{-z}| = e^{-\mathrm{Re}(z)} \leq e^{|z|}$$

hence $\sigma = 1$ is an exponential bound of f_0. Conversely, if $|f_0(z)| \leq \kappa e^{\sigma|z|}$ holds for any complex number z, then

$$\forall\, x \geq 0,\ f_0(-x) = e^x \leq \kappa e^{\sigma|-x|} = \kappa e^{\sigma x},$$

thus $\sigma \geq 1$. Finally, $\sigma(f_0) = 1$.

3. If for some $\kappa > 0$ and $\sigma \geq 0$, $\forall\, n \in \mathbb{N}$, $|n! a_n| \leq \kappa \sigma^n$, then

$$|f(z)| \leq \kappa \sum_{n=0}^{+\infty} \frac{(\sigma|z|)^n}{n!} = \kappa e^{\sigma|z|}$$

hence f is of exponential type and $\sigma(f) \leq \sigma$.

4. For any $n \in \mathbb{N}$ and $r > 0$,

$$a_n = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}},$$

hence if $\forall\, z \in \mathbb{C}$, $|f(z)| \leq \kappa e^{\sigma|z|}$, by the M-L inequality,

$$|a_n| \leq \frac{1}{2\pi} \kappa \frac{e^{\sigma r}}{r^{n+1}} \times 2\pi r = \kappa \frac{e^{\sigma r}}{r^n}.$$

If $n = 0$, the infimum of the right-hand side with respect to $r > 0$ is κ. Otherwise, the right-hand side is a differentiable function of $r > 0$ that tends to $+\infty$ when $r \to 0$ or $r \to +\infty$. Therefore, it has a minimum at a $r > 0$ that satisfies

$$\frac{d}{dr} \ln \left(\kappa \frac{e^{\sigma r}}{r^n} \right) = 0.$$

Given that

$$\frac{d}{dr}\ln\left(\kappa\frac{e^{\sigma r}}{r^n}\right) = \frac{d}{dr}\left(\ln\kappa + \sigma r - n\ln r\right) = \sigma - \frac{n}{r},$$

we have

$$|a_n| \leq \min_{r>0} \kappa\frac{e^{\sigma r}}{r^n} = \kappa\left(\frac{e\sigma}{n}\right)^n.$$

5. Let $\sigma > \sigma(f)$. Let $\lambda < 1$ such that $\lambda\sigma > \sigma(f)$; there is a $\kappa > 0$ such that

$$\forall\, z \in \mathbb{C},\ |f(z)| \leq \kappa e^{\lambda\sigma|z|}.$$

The result of the previous question yields for any $n > 0$

$$|a_n| \leq \kappa\left(\frac{e\lambda\sigma}{n}\right)^n$$

and $|a_0| \leq \kappa$, therefore

$$\forall\, n \in \mathbb{N}^*,\ |n!a_n| \leq \kappa\left[\sqrt{2\pi n}\lambda^n(1+\epsilon_n)\right]\sigma^n \ \text{ with } \lim_{n\to+\infty}\epsilon_n = 1.$$

The sequence

$$\left|\kappa\left[\sqrt{2\pi n}\lambda^n(1+\epsilon_n)\right]\right|,\ n \in \mathbb{N}^*$$

converges to 0 when n tends to $+\infty$ and thus has some upper bound κ' that we may select greater than κ. Finally, $|n!a_n| \leq \kappa'\sigma^n$ for any $n \in \mathbb{N}$.

6. The sum $\sum_{n=0}^{+\infty}|a_n|z^n$ is convergent for any $z \in \mathbb{C}$ because the Taylor expansion of f is absolutely convergent on the whole complex plane; the function

$$g : z \in \mathbb{C} \mapsto \sum_{n=0}^{+\infty}|a_n|z^n$$

is actually an entire function.

For any σ such that $\sigma(f) < \sigma$, by question 5, there is a $\kappa > 0$ such that $|n!a_n| \leq \kappa\sigma^n$ and thus $|n!|a_n|| \leq \kappa\sigma^n$. By question 3, the function g is of exponential type and $\sigma(g) \leq \sigma$; thus $\sigma(g) \leq \sigma(f)$. The converse inequality may be proved by the same kind of argument (or directly as $|f(z)| \leq g(|z|)$).

7. For any $\sigma \in \mathbb{R}$, $e^{-\sigma t}f_0(t) = e^{-(\sigma+1)t}$, hence the function $t \in \mathbb{R}_+ \mapsto e^{-\sigma t}f_0(t)$ is integrable if and only if $\sigma > -1$. Therefore, the Laplace transform of the function f_0 is defined exactly on $\{s \in \mathbb{C} \mid \text{Re}\ s > -1\}$. For any such s, we have

$$\mathcal{L}[f_0](s) = \int_0^{+\infty} e^{-t}e^{-st}\,dt = \left[-\frac{e^{-(s+1)t}}{s+1}\right]_0^{+\infty} = \frac{1}{s+1}.$$

In general, if f is an entire function of exponential type, then for any complex number s such that $\sigma = \mathrm{Re}(s) > \sigma(f)$, there is a $\epsilon > 0$ such that $\sigma - \epsilon > \sigma(f)$, hence there is some $\kappa > 0$ such that

$$\forall\, t > 0,\ |f(t)e^{-st}| = |f(t)|e^{-\sigma t} \leq \kappa e^{-\epsilon t},$$

thus the function $t \in \mathbb{R}_+ \mapsto f(t)e^{-st}$ is integrable.

8. If $\mathrm{Re}(s) > \sigma(f)$, we have

$$\mathcal{L}[f](s) = \int_{\mathbb{R}_+} \left[\sum_{n=0}^{+\infty} a_n t^n\right] e^{-st}\, dt.$$

For any $m \in \mathbb{N}$,

$$\left|\left[\sum_{n=0}^{m} a_n t^n\right] e^{-st}\right| \leq \left[\sum_{n=0}^{+\infty} |a_n| t^n\right] e^{-(\mathrm{Re}(s))t} = g(t)e^{-(\mathrm{Re}(s))t}.$$

Therefore, as $\sigma(g) = \sigma(f)$, for any $\sigma > 0$ such that $\sigma(f) < \sigma < \mathrm{Re}(s)$, there is a $\kappa > 0$ such that

$$\left|\left[\sum_{n=0}^{m} |a_n| t^n\right] e^{-st}\right| \leq \kappa e^{\sigma t} e^{-(\mathrm{Re}(s))t} = \kappa e^{-(\mathrm{Re}(s)-\sigma)t}.$$

The right-hand side of this inequality is an integrable function of t; by the dominated convergence theorem

$$\mathcal{L}[f](s) = \int_{\mathbb{R}_+} \left[\sum_{n=0}^{+\infty} a_n t^n\right] e^{-st}\, dt = \sum_{n=0}^{+\infty} a_n \left[\int_{\mathbb{R}_+} t^n e^{-st}\, dt\right].$$

For any natural number n, we have

$$\int_{\mathbb{R}_+} t^n e^{-st}\, dt = \mathcal{L}[t \mapsto t^n](s) = \frac{n!}{s^{n+1}}$$

and therefore

$$\mathcal{L}[f](s) = \sum_{n=0}^{+\infty} \frac{n! a_n}{s^{n+1}}.$$

Now, the right-hand side of this equation is a Laurent series centered on the origin which is convergent whenever $\mathrm{Re}(s) > \sigma(f)$. Consequently, the sum $g(s)$ is actually convergent and holomorphic in the annulus $A(0, \sigma(f), +\infty)$.

Now, the set

$$\Omega = \mathrm{dom}\, \mathcal{L}[f] \cup A(0, \sigma(f), +\infty).$$

is open and connected as the union of two connected sets with a non-empty intersection. We may define on Ω the function

$$\overline{\mathcal{L}}[f](s) = \left| \begin{array}{ll} \mathcal{L}[f](s) & \text{if } s \in \mathrm{dom}\, \mathcal{L}[f], \\ g(s) & \text{if } |s| > \sigma(f). \end{array} \right.$$

This definition is non-ambiguous: since $\mathcal{L}[f](s)$ and g are holomorphic and identical on the non-empty set $\{s \in \mathbb{C} \mid \operatorname{Re} s > \sigma(f)\}$, by the isolated zeros theorem, they have the same values on the connected open set

$$\operatorname{dom} \mathcal{L}[f] \cap A(0, \sigma(f), +\infty).$$

The function $\overline{\mathcal{L}}[f]$ clearly is a holomorphic extension of $\mathcal{L}[f]$ to Ω. This extension is unique: the set Ω is connected as the union of two connected sets with a non-empty intersection, therefore the isolated zeros theorem also applies.

For $f = f_0$, we have

$$\operatorname{dom} \mathcal{L}[f] = \{s \in \mathbb{C} \mid \operatorname{Re}(s) > -1\}$$

and as $\sigma(f_0) = 1$,

$$A(0, \sigma(f_0), +\infty) = \{s \in \mathbb{C} \mid |s| > 1\},$$

thus $\Omega = \mathbb{C} \setminus \{-1\}$. The function

$$s \in \mathbb{C} \setminus \{-1\} \mapsto \frac{1}{s+1}$$

is holomorphic. As it extends the original Laplace transform of f_0, this is the extended Laplace transform of f_0.

9. For any $t \geq 0$ and any $r > \sigma(f)$,

$$\frac{1}{i2\pi} \int_{r[\circlearrowleft]} \overline{\mathcal{L}}[f] e^{st}\, ds = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \left[\sum_{n=0}^{+\infty} \frac{n! a_n}{s^{n+1}} \right] e^{st}\, ds.$$

The series $\sum_{n=0}^{+\infty} n! a_n / s^{n+1}$ is convergent, uniformly with respect to s on the circle of radius r centered on the origin, hence

$$\frac{1}{i2\pi} \int_{r[\circlearrowleft]} \overline{\mathcal{L}}[f] e^{st}\, ds = \sum_{n=0}^{+\infty} n! a_n \left[\frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{e^{st}}{s^{n+1}}\, ds \right].$$

The function $s \in \mathbb{C}^* \mapsto e^{st}/s^{n+1}$ is complex-differentiable, therefore

$$\frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{e^{st}}{s^{n+1}}\, ds = \operatorname{res}\left(s \mapsto \frac{e^{st}}{s^{n+1}}, 0 \right).$$

Since

$$\lim_{s \to 0} s^{n+1} \frac{e^{st}}{s^{n+1}} = 1,$$

the origin is a pole of order $n+1$ of this function and

$$\operatorname{res}\left(s \mapsto \frac{e^{st}}{s^{n+1}}, 0 \right) = \lim_{s \to 0} \frac{1}{n!} \frac{d^n}{ds^n} e^{st} = \frac{t^n}{n!}.$$

Finally,

$$\frac{1}{i2\pi}\int_{r[\circlearrowleft]}\overline{\mathcal{L}}[f]e^{st}\,ds=\sum_{n=0}^{+\infty}n!a_n\frac{t^n}{n!}=\sum_{n=0}^{+\infty}a_nt^n=f(t).$$