

# Complex-Differentiability

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## Core Definitions

**Definition – Complex-Differentiability & Derivative.** Let  $f : A \subset \mathbb{C} \rightarrow \mathbb{C}$ . The function  $f$  is *complex-differentiable* at an interior point  $z$  of  $A$  if the *derivative* of  $f$  at  $z$ , defined as the limit of the difference quotient

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

exists in  $\mathbb{C}$ .

**Remark – Why Interior Points?** The point  $z$  is an interior point of  $A$  if

$$\exists r > 0, \forall h \in \mathbb{C}, |h| < r \rightarrow z + h \in A.$$

In the definition above, this assumption ensures that  $f(z+h)$  – and therefore the difference quotient – are well defined when  $|h|$  is (nonzero and) small enough. Therefore, the derivative of  $f$  at  $z$  is defined as the limit in “all directions at once” of the difference quotient of  $f$  at  $z$ . To question the existence of the derivative of  $f : A \subset \mathbb{C} \rightarrow \mathbb{C}$  at every point of its domain, we therefore require that every point of  $A$  is an interior point, or in other words, that  $A$  is open.

**Definition – Holomorphic Function.** Let  $\Omega$  be an open subset of  $\mathbb{C}$ . A function  $f : \Omega \rightarrow \mathbb{C}$  is *complex-differentiable* – or *holomorphic* – in  $\Omega$  if it is complex-differentiable at every point  $z \in \Omega$ . If additionally  $\Omega = \mathbb{C}$ , the function is *entire*.

**Examples – Elementary Functions.**

1. Every constant function  $f : z \in \mathbb{C} \mapsto \lambda \in \mathbb{C}$  is holomorphic as

$$\forall z \in \mathbb{C}, f'(z) = \lim_{h \rightarrow 0} \frac{\lambda - \lambda}{h} = 0.$$

2. The identity function  $f : z \in \mathbb{C} \mapsto z$  is holomorphic:

$$\forall z \in \mathbb{C}, f'(z) = \lim_{h \rightarrow 0} \frac{(z+h) - z}{h} = 1.$$

3. The inverse function  $f : z \in \mathbb{C}^* \rightarrow 1/z$  is holomorphic: the set  $\mathbb{C}^*$  is open and for any  $z \in \mathbb{C}^*$  and any  $h \in \mathbb{C}$  such that  $z+h \neq 0$ , we have

$$\frac{1/(z+h) - 1/z}{h} = -\frac{1}{z(z+h)},$$

hence

$$f'(z) = \lim_{h \rightarrow 0} -\frac{1}{z(z+h)} = -\frac{1}{z^2}.$$

4. The complex conjugate function  $f : z \in \mathbb{C} \rightarrow \bar{z}$  is nowhere complex-differentiable. Its difference quotient satisfies

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{h}}{h},$$

therefore when  $t \in \mathbb{R}$ ,

$$\lim_{t \rightarrow 0} \frac{f(z+t) - f(z)}{t} = +1,$$

but

$$\lim_{t \rightarrow 0} \frac{f(z+it) - f(z)}{it} = -1,$$

hence the difference quotient has no limit when  $h \rightarrow 0$ .

**Remark – Holomorphy & Continuity.** Holomorphic functions are continuous; this property is unsurprising and its proof is straightforward. But actually, derivatives of holomorphic functions are also continuous! This may come as a surprise since the corresponding statement doesn't hold in the context of real analysis: there are some real-valued functions of a real variable that are differentiable and whose derivative is *not* continuous<sup>1</sup>.

<sup>1</sup>In the context of real analysis, derivatives can't be totally arbitrary either. They satisfy for example the intermediate value theorem (a property which is weaker than continuity). Refer to (Freiling 1999) for a complete characterization.

We mention this property now because we will use it to simplify the statements of some results of the current and subsequent chapters. Unfortunately, we cannot prove it yet; it is a consequence of Cauchy’s integral theory which is far from being trivial. To make sure that we won’t develop a circular argument, we flag the results that use this property with the symbol [†], until we can prove it.

**Remark – Historical Perspective.** We should not feel too bad about the (temporary) assumption that the derivative is continuous; after all, it was good enough for the best mathematicians of the 19th century.

A major result of complex analysis, Cauchy’s integral theorem, was originally formulated under the assumption that the derivative exists and is continuous<sup>2</sup> (Cauchy 1825). We have to wait for the paper “Sur la définition générale des fonctions analytiques, d’après Cauchy” (Goursat 1900) to officially get rid of this assumption:

J’ai reconnu depuis longtemps que la démonstration du théorème de Cauchy, que j’ai donnée en 1883, ne supposait pas la continuité de la dérivée. Pour répondre au désir qui m’a été exprimé par M. le Professeur W. F. Osgood, je vais indiquer ici rapidement comment on peut faire cette extension.

which means:

I have long recognized that the proof of Cauchy’s theorem, that I have given in 1883, did not assume the continuity of the derivative. To meet the desire which was expressed to me by Professor W. F. Osgood, I’ll tell here quickly how we can make this extension.

Because of this improvement, Cauchy’s integral theorem is also known as the “Cauchy-Goursat theorem”. Refer to (Hille 1973, footnote p.163) for a broader historical perspective on this subject.

## Derivative and Complex-Differential

**Definition – Real/Complex Vector Space.** A vector space is *real* or *on*  $\mathbb{R}$  if its field of scalars is the real line; it is *complex* or *on*  $\mathbb{C}$  if its field of scalars is the complex plane.

**Remark – Complex Vector Spaces are Real Vector Spaces.** If the set  $E$  is endowed with a structure of complex vector space, it is automatically endowed with a structure of real vector space. Because of this ambiguity, it may be necessary to qualify the usual concepts of linear algebra – for example to say

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<sup>2</sup>To be honest, this assumption is only implicit, but you can’t really blame Cauchy for this lack of precision. The standards of quality for Mathematics in the 19th century were quite different from the present ones.

that a function is *real-linear* or *complex-linear* instead of simply linear – to be totally explicit about the structure to which we refer.

**Example – The Complex Plane.** The set  $\mathbb{C}$  is a complex vector space with the sum

$$(x + iy) + (v + iw) = (x + v) + i(y + w)$$

and scalar-vector multiplication

$$(\mu + i\nu)(x + iy) = (\mu x - \nu y) + i(\mu y + \nu x)$$

It is of dimension 1 with for example  $\{1\}$  (the single vector  $1 \in \mathbb{C}$ ) as a basis; indeed every complex number  $z \in \mathbb{C}$  is a linear combination of the vectors of  $\{1\}$  (as  $z = z1$ ), and the vectors of  $\{1\}$  are linearly independent (the only scalar  $\lambda \in \mathbb{C}$  such that  $\lambda 1 = 0$  is  $\lambda = 0$ ).

Note how things change if we consider the plane as a real vector space: it is of dimension 2 with for example  $\{1, i\}$  as a basis. In particular, the vectors 1 and  $i$  which are complex-colinear are not real-colinear.

**Definition – Complex-Linearity.** Let  $E$  and  $F$  be complex normed vector spaces. A function  $\ell : E \rightarrow F$  is *complex-linear* if it is additive and *complex-homogeneous*:

$$\forall u \in E, \forall v \in E, \ell(u + v) = \ell(u) + \ell(v),$$

$$\forall \lambda \in \mathbb{C}, \forall u \in E, \ell(\lambda u) = \lambda \ell(u).$$

**Definition – Complex-Differential.** Let  $f : A \subset E \rightarrow F$  where  $E$  and  $F$  are complex normed vector spaces. Let  $z$  be an interior point of  $A$ ; the *complex-differential* of  $f$  at  $z$  is a complex-linear continuous operator  $df_z : E \rightarrow F$  such that

$$\lim_{h \rightarrow 0} \frac{\|f(z + h) - f(z) - df_z(h)\|}{\|h\|} = 0,$$

or equivalently in expanded form

$$f(z + h) = f(z) + df_z(h) + \epsilon_z(h)\|h\| \quad \text{with} \quad \lim_{h \rightarrow 0} \epsilon_z(h) = \epsilon_z(0) = 0.$$

If such an operator exists, it is unique.

**Remark – Real/Complex-Differentiability.** Since the complex vector spaces  $E$  and  $F$  are real vector spaces, we may also use the classic concept of differential from Real Analysis for the function  $f : E \rightarrow F$ . We call this operator *real-differential* to avoid any ambiguity with the complex-differential. The definitions of both operators are identical, except that the complex-differential is required to be a complex-linear operator when the real-differential is only required to be real-linear.

**Proof – Uniqueness.** If for small values of  $h$  the function  $f$  satisfies

$$f(z + h) = f(z) + \ell_z^1(h) + \epsilon_z^1(h)\|h\| = f(z) + \ell_z^2(h) + \epsilon_z^2(h)\|h\|$$

with continuous linear operators  $\ell_z^1$  and  $\ell_z^2$  and functions  $\epsilon_z^1$  and  $\epsilon_z^2$  such that

$$\lim_{h \rightarrow 0} \epsilon_z^1(h) = \epsilon_z^1(0) = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \epsilon_z^2(h) = \epsilon_z^2(0) = 0,$$

then for any  $u \in E$

$$(\ell_z^1 - \ell_z^2)(u) = \lim_{t \rightarrow 0} \frac{(\ell_z^1 - \ell_z^2)(tu)}{t} = \lim_{t \rightarrow 0} (\epsilon_z^2(tu) - \epsilon_z^1(tu)) \|u\| = 0$$

and consequently  $\ell_z^1 = \ell_z^2$ . ■

**Theorem – Derivative and Differential.** Let  $f : A \rightarrow \mathbb{C}$  with  $A \subset \mathbb{C}$  and let  $z$  be an interior point of  $A$ . The complex-differential  $df_z$  exists if and only if the derivative  $f'(z)$  exists. In this case, we have

$$\forall h \in \mathbb{C}, df_z(h) = f'(z)h.$$

**Proof.** If  $f'(z)$  exists, the mapping  $h \in \mathbb{C} \mapsto f'(z)h$  is complex-linear and

$$\lim_{h \rightarrow 0} \frac{|f(z+h) - f(z) - f'(z)h|}{|h|} = \lim_{h \rightarrow 0} \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| = 0,$$

hence, it is the differential of  $f$  at  $z$ . Conversely, if  $df_z$  exists, its complex-linearity yields  $df_z(h) = df_z(1)h$ . Therefore,

$$\lim_{h \rightarrow 0} \left| \frac{f(z+h) - f(z)}{h} - df_z(1) \right| = \lim_{h \rightarrow 0} \frac{|f(z+h) - f(z) - df_z(h)|}{|h|} = 0$$

thus  $f'(z)$  exists and is equal to  $df_z(1)$ . ■

## Calculus

**Theorem – Sum and Product Rules.** Let  $f : A \rightarrow \mathbb{C}$  and  $g : A \rightarrow \mathbb{C}$  with  $A \subset \mathbb{C}$  and let  $z$  an interior point of  $A$ . If  $f$  and  $g$  are complex-differentiable at  $z$ , the derivative of  $f + g$  at  $z$  exists and

$$(f + g)'(z) = f'(z) + g'(z),$$

the derivative of  $f \times g$  at  $z$  exists and

$$(f \times g)'(z) = f'(z) \times g(z) + f(z) \times g'(z).$$

**Proof.** For any  $h \in \mathbb{C}$  such that  $z + h \in A$ , we have

$$\frac{(f + g)(z + h) - (f + g)(z)}{h} = \frac{f(z + h) - f(z)}{h} + \frac{g(z + h) - g(z)}{h},$$

hence the derivative of  $f + g$  at  $z$  exists and satisfies the sum rule. On the other hand,

$$\begin{aligned} \frac{(f + g)(z + h) - (f + g)(z)}{h} &= \\ &= \frac{f(z + h) - f(z)}{h}g(z) + f(z + h)\frac{g(z + h) - g(z)}{h}, \end{aligned}$$

hence the derivative of  $f \times g$  exists and satisfies the product rule.  $\blacksquare$

**Theorem – Chain Rule.** Let  $f : A \rightarrow \mathbb{C}$ ,  $g : B \rightarrow \mathbb{C}$  with  $A, B$  two subsets of  $\mathbb{C}$ . If  $z$  is an interior point of  $A$ ,  $f$  is complex-differentiable at  $z$ ,  $f(z)$  is an interior point of  $B$  and  $g$  is complex-differentiable at  $f(z)$ , then  $g \circ f$  is complex-differentiable at  $z$  and

$$(g \circ f)'(z) = g'(f(z)) \times f'(z).$$

**Proof.** Given the assumption, we have for  $h$  small enough

$$f(z + h) - f(z) = f'(z)h + \epsilon_z^1(h)|h|$$

and

$$g(f(z) + h) - g(f(z)) = g'(f(z))h + \epsilon_{f(z)}^2(h)|h|$$

with

$$\lim_{h \rightarrow 0} \epsilon_z^1(h) = \epsilon_z^1(0) = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \epsilon_{f(z)}^2(h) = \epsilon_{f(z)}^2(0) = 0.$$

Consequently,

$$\begin{aligned} g(f(z + h)) - g(f(z)) &= g'(f(z))(f(z + h) - f(z)) \\ &\quad + \epsilon_{f(z)}^2(f(z + h) - f(z))|f(z + h) - f(z)|, \end{aligned}$$

which can be expanded into

$$g(f(z + h)) - g(f(z)) = g'(f(z))f'(z)h + \epsilon_z^3(h)|h|,$$

where

$$\epsilon_z^3(h) = g'(f(z))\epsilon_z^1(h) + \epsilon_{f(z)}^2(f(z + h) - f(z)) \left| f'(z) \frac{h}{|h|} + \epsilon_z^1(h) \right|$$

and satisfies

$$\lim_{h \rightarrow 0} \epsilon_z^3(h) = \epsilon_z^3(0) = 0.$$

This decomposition proves the existence of the complex-differential of  $g \circ f$  at  $z$  as well as the equality  $(g \circ f)'(z) = g'(f(z))f'(z)$ .  $\blacksquare$

**Corollary – Quotient Rule.** Let  $f : A \rightarrow \mathbb{C}$  and  $g : A \rightarrow \mathbb{C}$  with  $A \subset \mathbb{C}$  and let  $z$  be an interior point of  $A$  such that  $g(z) \neq 0$ . If  $f$  and  $g$  are complex-differentiable at  $z$ , then  $f/g$  is complex-differentiable at  $z$  and

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

**Proof.** By the chain rule applied to the function  $g$  and  $z \mapsto 1/z$ , the derivative of  $1/g$  is  $-g'/g^2$ . The desired result then follows from the product rule. ■

**Examples – Polynomials & Rational Functions.** Any polynomial  $p$  with complex coefficients

$$p : z \in \mathbb{C} \mapsto a_0 + a_1z + \cdots + a_nz^n$$

is holomorphic on  $\mathbb{C}$  since it is the sum of products of holomorphic functions. By the quotient rule, the quotient of two polynomials  $p$  and  $q$  – with a non-zero  $q$  – is also holomorphic on the open set  $\{z \in \mathbb{C} \mid q(z) \neq 0\}$ .

## Cauchy-Riemann Equations

It is sometimes convenient to remember that the set  $\mathbb{C}$  is only  $\mathbb{R}^2$ , or in other words that we can always identify the complex number  $z = x + iy$  with the pair of real numbers  $(x, y)$ .

For complex-valued functions of a complex variable, we can perform this identification for the variables  $z = x + iy$  and/or for the values  $f = u + iv$ . Both options are actually interesting and lead to slightly different characterizations of holomorphic functions.

**Theorem – Cauchy-Riemann Equations.** Let  $\Omega$  be an open subset of  $\mathbb{C}$ . The function  $f = u + iv : \Omega \rightarrow \mathbb{C}$  is complex-differentiable on  $\Omega$  if and only if:

- it is real-differentiable on  $\Omega$  and
- for every  $z$  in  $\Omega$ ,  $df_z$  is complex-linear.

The second clause may be replaced by any of the following:

1. the function  $f$  satisfies

$$\forall z \in \Omega, df_z(i) = idf_z(1),$$

2. the function  $f$  satisfies the (*complex*) *Cauchy-Riemann equation*:

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y},$$

3. the functions  $u$  and  $v$  satisfy the (scalar) Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If the complex-differentiability holds, we have

$$\begin{aligned} f' = (z \mapsto d_z f(1)) &= \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \\ &= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \end{aligned}$$

**Remark – A Geometric Insight.** We may rewrite the scalar Cauchy-Riemann equations as

$$\begin{bmatrix} \partial v/\partial x \\ \partial v/\partial y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}.$$

This formula provides the following insight: the gradient of the imaginary part of a holomorphic function is obtained by a rotation of  $\pi/2$  of the gradient of its real part.

**Proof.** By definition,  $f$  is complex-differentiable if and only if it is real-differentiable and its differential is complex-linear.

Assume that  $f$  is real-differentiable; the real-differential  $\ell = df_z$  is real-linear, that is, additive and real-homogeneous. If additionally  $\ell(i) = i\ell(1)$ , then we have for any real numbers  $\mu, \nu, x$  and  $y$

- $\ell(\mu x) = \mu\ell(x)$ ,
- $\ell(i\nu x) = \nu x\ell(i) = \nu x i\ell(1) = i\nu\ell(x)$ ,
- $\ell(i\mu y) = \mu\ell(iy)$ ,
- $\ell(-\nu y) = -\nu y\ell(1) = i^2\nu y\ell(1) = i\nu y\ell(i) = i\nu\ell(iy)$ .

The function  $\ell$  is additive, hence  $\ell((\mu + i\nu)(x + iy)) = (\mu + i\nu)\ell(x + iy)$ : the function  $\ell$  is complex-homogeneous and therefore complex-linear. Hence, property 1 yields the complex-linearity of the differential.

As additionally

$$\frac{\partial f}{\partial x}(z) = df_z(1), \quad \frac{\partial f}{\partial y}(z) = df_z(i),$$

properties 1 and 2 are equivalent.

The function  $f = (u, v) = u + iv$  is real-differentiable if and only if  $u$  and  $v$  are real-differentiable. In this case,  $df = (du, dv) = du + idv$ , hence

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$



which yields the equivalence between properties 2 and 3. ■

There is a variant of this theorem that does not require to check explicitly for the existence of the real-differential:

**Corollary – Cauchy-Riemann Equations (Alternate)** [†]. Let  $\Omega$  be an open subset of  $\mathbb{C}$ . The function  $f = u + iv : \Omega \rightarrow \mathbb{C}$  is complex-differentiable in  $\Omega$  if and only if any of the following conditions holds:

1. The partial derivatives  $\partial f/\partial x$  and  $\partial f/\partial y$  exist, are continuous and

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

2. The partial derivatives  $\partial u/\partial x$ ,  $\partial u/\partial y$ ,  $\partial v/\partial x$  and  $\partial v/\partial y$  exist, are continuous and

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

**Proof.** If the partial derivatives of  $f$  exist and are continuous, or equivalently the partial derivatives of  $u$  and  $v$  exist and are continuous, then  $f$  is continuously real-differentiable and we can apply the previous theorem to get our conclusion.

Reciprocally, if the derivative of  $f$  exist, then it is continuous, hence the partial derivatives of  $f$  (or of  $u$  and  $v$ ) are continuous. The previous theorem also shows that the Cauchy-Riemann equations are satisfied. ■

**Definition & Example – Exponential.** The *exponential* of a complex number  $z = x + iy$  is the complex number denoted  $e^z$  or  $\exp z$  defined by

$$e^z = e^x \times (\cos y + i \sin y).$$

The exponential function  $\exp : \mathbb{C} \rightarrow \mathbb{C}$  satisfies

$$\frac{\partial e^{x+iy}}{\partial x} = e^x \times (\cos y + i \sin y) = e^{x+iy}$$

and

$$\frac{1}{i} \frac{\partial e^{x+iy}}{\partial y} = \frac{1}{i} e^x \times (-\sin y + i \cos y) = e^{x+iy}.$$

Both partial derivatives exist and are continuous. Since they also satisfy the Cauchy-Riemann equations, the exponential function is complex-differentiable and

$$\frac{de^z}{dz} = e^z.$$

**Definition & Example – Logarithm.** The *principal value of the logarithm* is the function  $\log : \mathbb{C} \setminus \mathbb{R}_- \rightarrow \mathbb{C}$  defined by

$$\log r e^{i\theta} = (\ln r) + i\theta, \quad r > 0, \quad \theta \in ]-\pi, \pi[.$$

It is a bijection from  $\mathbb{C} \setminus \mathbb{R}_-$  into  $\mathbb{R} \times ]-\pi, +\pi[$  and for every  $z \in \mathbb{C} \setminus \mathbb{R}_-$

$$e^{\log z} = \exp \circ \log(z) = z.$$

The exponential function is continuously real-differentiable and

$$d \exp_z(h) = (\exp z) \times h$$

hence its differential  $d \exp_z$  is invertible. By the inverse function theorem,  $\log$  is real-differentiable on  $\mathbb{C} \setminus \mathbb{R}_-$  and

$$d \exp_{\log z} \circ d \log_z(h) = z \times d \log_z(h) = h.$$

Consequently,  $d \log_z(h) = h/z$ , which is a complex-linear function of  $h$ . Hence,  $\log$  is complex-differentiable and

$$\log'(z) = \frac{1}{z}.$$

**Definition & Example – Power Functions.** The *principal value of the power with exponent  $\alpha$*  – where  $\alpha \in \mathbb{C}$  – is the function  $z \in \mathbb{C} \setminus \mathbb{R}_- \mapsto z^\alpha \in \mathbb{C}$  defined by

$$z^\alpha = e^{\alpha \log z}.$$

By the chain rule, it is holomorphic and

$$\begin{aligned} \frac{dz^\alpha}{dz} &= e^{\alpha \log z} \times \frac{d\alpha \log z}{dz} \\ &= \alpha e^{\alpha \log z} e^{-\log z} \\ &= \alpha z^{\alpha-1} \end{aligned}$$

## Appendix – Terminology and Notation

It is common to use of the word “holomorphic” and the notation  $\mathcal{H}(\Omega)$  to refer to functions that are complex-differentiable on some open set  $\Omega$ ; it is for example the convention of the classic “Real and Complex Analysis” book (Rudin 1987).

The term “holomorphic” appears in “Théorie des fonctions elliptiques” (1875), by Charles Briot & Claude Bouquet, two students of Augustin-Louis Cauchy:

Lorsqu’une fonction est continue, monotrope, et a une dérivée, quand la variable se meut dans une certaine partie du plan, nous dirons qu’elle est *holomorphe* dans cette partie du plan. Nous indiquons par cette dénomination qu’elle est semblable aux fonctions entières qui jouissent de cette propriété dans toute l’étendue du plan.

In essence, a function is holomorphic if it is continuous, single-valued and differentiable in a subset of the complex plane. The prefix “holo” (from ancient Greek) means “entire”; it makes sense because such a function is similar to polynomials, which have these properties in the full complex plane and were called “entire functions” in the 19th century<sup>3</sup>.

The most common alternate notations and terms used in the literature to refer to holomorphic functions probably are:

- $\mathcal{A}(\Omega)$ . The symbol “A” refers to the term “analytic”; it is often used interchangeably with the term “holomorphic”. Originally, “analytic” means “locally defined as a power series”, but both concepts actually refer to the same class of functions. This is a classic – but not trivial – result of complex analysis.
- $C^\omega(\Omega)$ . Another result of the theory of analytic functions: analytic functions are “more than smooth”: they all belong to the set  $C^\infty(\Omega)$  of smooth functions, but not every smooth function is analytic. Hence, it makes sense to use the symbol  $\omega$  – that denotes the smallest infinite ordinal number – as an exponent.
- $\mathcal{O}(\Omega)$  (used e.g. in “Theory of Complex Functions” by Remmert (1991)). Jean-Pierre Demailly (2009) traces the origin of this notation to the word “olomorfico” (“holomorphic” in Italian), but Hans Grauert and Reinhold Remmert (1984) have a different analysis: the symbol “O” may have been chosen by Henri Cartan, which is quoted saying (in French) that:

Je m'étais simplement inspiré d'une notation utilisée par van der Waerden dans son classique traité “Moderne Algebra” (cf. par exemple §16 de la 2e édition allemande, p.52)

which means

I simply took inspiration from a notation used by van der Waerden in his classic treatise “Modern Algebra” (see e.g. §16 of the 2nd german edition, p.52)

If this interpretation is correct, then the symbol “O” probably comes from the word “ordnung” (“order” in German).

## References

Briot, Charles, and Jean-Claude Bouquet. 1875. *Théorie des fonctions elliptiques*. 2nd ed. Gauthier-Villars, Paris. <http://gallica.bnf.fr/ark:>

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<sup>3</sup>Nowadays, a function is entire if it is defined and holomorphic on  $\mathbb{C}$ , which is a more consistent definition. Polynomials are still entire functions, but they are not the only ones.

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Cauchy, Augustin-Louis. 1825. *Mémoire sur les intégrales définies, prise entre des limites imaginaires*. Chez de Bure Frères. doi:10.3931/e-rara-26300.

Demailly, Jean-Pierre. 2009. *Fonctions holomorphes et surfaces de riemann*. [https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/variable\\_complexe.pdf](https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/variable_complexe.pdf).

Freiling, Chris. 1999. “On the Problem of Characterizing Derivatives.” *Real Analysis Exchange* 23(2): 805–12. <http://projecteuclid.org/euclid.rae/1337001388>.

Goursat, Edouard. 1900. “Sur La Définition Générale Des Fonctions Analytiques d’après Cauchy.” *Transactions of the American Mathematical Society*. 1. American Mathematical Society (AMS), Providence, RI.: 14–16. doi:10.2307/1986398.

Grauert, Hans, and Reinhold Remmert. 1984. *Coherent Analytic Sheaves*. *Grundlehren Der Mathematischen Wissenschaften*. Springer-Verlag. XVIII, 249 p. DM 118.00; \\\$ 46.30 (1984).?

Hille, Einar. 1973. *Analytic Function Theory. Vol. I, II*. 2nd ed. corrected. Chelsea Publishing Company.

Remmert, Reinhold. 1991. *Theory of Complex Functions. Translation from the German by Robert B. Burckel*. Springer-Verlag.

Rudin, Walter. 1987. *Real and Complex Analysis. 3rd Ed.* 3rd ed. McGraw-Hill.