# Cauchy's Integral Theorem – Local Version

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## Exercises

### A Fourier Transform

We wish to compute for any real number $\omega$ the value of the integral

$$\hat{x}(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-i\omega t} \, dt$$

when $x : \mathbb{R} \to \mathbb{R}$ is the Gaussian function defined by

$$\forall \, t \in \mathbb{R}, \; x(t) = e^{-t^2/2}.$$

We remind you of the value of the *Gaussian integral* (see e.g. Wikipedia):

$$\hat{x}(0) = \int_{-\infty}^{+\infty} e^{-t^2/2} \, dt = \sqrt{2\pi}$$

**Questions**

1. Show that for any pair of real numbers $\tau$ and $\omega$, we can compute

$$\int_{-\tau}^{\tau} x(t) e^{-i\omega t}\, dt$$

from the line integral of a fixed holomorphic function on a path $\gamma$ that depends on $\tau$ and $\omega$.

2. Use Cauchy's integral theorem to evaluate $\hat{x}(\omega)$.

**Answers**

1. We may denote $x$ the extension to the complex plane of the original Gaussian function $x$, defined by:

$$\forall\, z \in \mathbb{C},\ x(z) = e^{-z^2/2}.$$

It is holomorphic as a composition of holomorphic functions. Let $\gamma = [-\tau \to \tau] + i\omega$. The line integral of $x$ along $\gamma$ satisfies

$$\int_\gamma x(z)\, dz = \int_0^1 x(-\tau(1-s) + \tau s + i\omega)\, (2\tau ds)$$

or, using the change of variable $t = -\tau(1-s) + \tau s$,

$$\int_\gamma x(z)\, dz = \int_{-\tau}^{\tau} x(t + i\omega)\, dt.$$

Since

$$x(t + i\omega) = e^{-(t+i\omega)^2/2} = e^{-t^2/2} e^{-i\omega t} e^{\omega^2/2},$$

we end up with

$$\int_\gamma x(z)\, dz = e^{\omega^2/2} \int_{-\tau}^{\tau} x(t) e^{-i2\pi f t}\, dt$$

2. Let $\nu = \tau + [0 \to i\omega]$; on the image of this path, we have

$$\forall\, s \in [0,1],\ |x(\nu(s))| = \left| e^{-(\tau + i\omega s)^2/2} \right| = e^{-\tau^2/2} e^{(\omega s)^2/2} \leq e^{-\tau^2/2} e^{\omega^2/2},$$

hence the M-L inequality provides

$$\left| \int_\mu x(z)\, dz \right| \leq (|\omega| e^{\omega^2/2}) e^{-\tau^2/2}$$

and thus,

$$\forall\, \omega \in \mathbb{R}, \lim_{|\tau| \to +\infty} \int_{\mu} x(z)\, dz = 0.$$

We may apply Cauchy's integral theorem to the function $x$ on the closed polyline

$$[-\tau + i\omega \;\to\; \tau + i\omega \;\to\; \tau \;\to\; -\tau \;\to\; -\tau + i\omega].$$

It is the concatenation of $\gamma = [-\tau \to \tau] + i\omega$, the reverse of $\mu_+ = \tau + [0 \to i\omega]$, the reverse of $\gamma_0 = [-\tau \to \tau]$ and finally $\mu_- = -\tau + [0 \to i\omega]$.

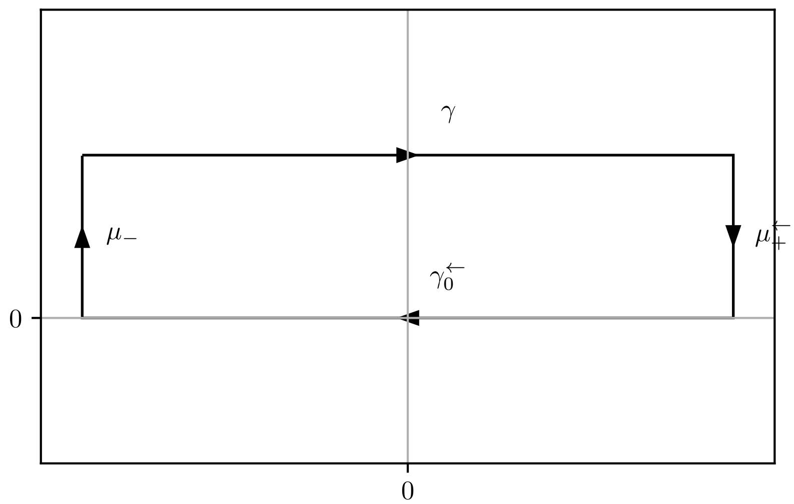


Figure 1: The closed path used in the application of Cauchy's integral theorem

The theorem provides

$$e^{\omega^2/2} \int_{-\tau}^{\tau} x(t) e^{-i\omega t}\, dt - \int_{\mu_+} x(z)\, dz - \int_{-\tau}^{\tau} x(t)\, dt + \int_{\mu_-} x(z)\, dz = 0.$$

When $\tau \to +\infty$, this equality yields

$$\int_{-\infty}^{\infty} x(t) e^{-i 2\pi f t}\, dt = \sqrt{2\pi} e^{-\omega^2/2}.$$

## Cauchy's Integral Formula for Disks

Let $\Omega$ be an open subset of the complex plane and $\gamma = c + r[\circlearrowleft]$. We assume that the closed disk $\overline{D}(c, r)$ is included in $\Omega$ (this is stronger than the requirement that $\gamma$ is a path of $\Omega$).

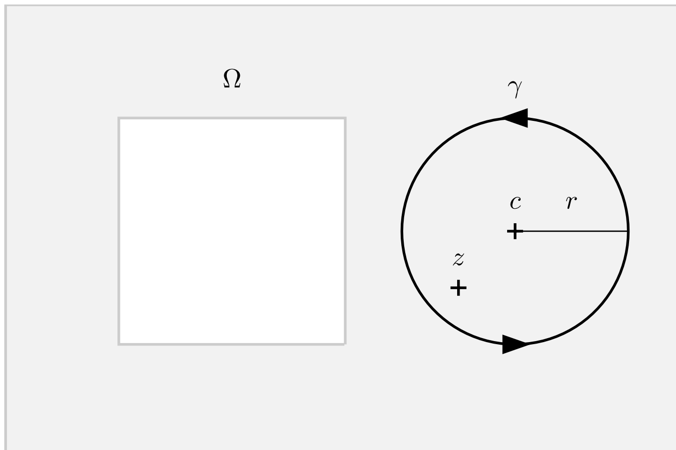


Figure 2: Geometry of Cauchy's integral formula for disks.

We wish to prove that for any holomorphic function $f : \Omega \to \mathbb{C}$,

$$\forall\, z \in \Omega,\ |z - c| < r \ \Rightarrow\ f(z) = \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w - z} dw.$$

**Questions**

1. What is the value of the line integral above when $|z - c| > r$?
2. Compute the line integral above when $z = c$ as an integral with respect to a real variable. When happens in this case when $r \to 0$ ?
3. Let $\epsilon > 0$ be such that $|z| + \epsilon < r$ and let $\lambda = z + \epsilon[\circlearrowleft]$. Provide two paths $\mu$ and $\nu$ whose images belong to (different) star-shaped subsets of $\Omega \setminus \{z\}$ and such that for any continuous function $g : \Omega \setminus \{z\} \to \mathbb{C}$,
$$\int_\gamma g(w)\, dw = \int_\lambda g(w) dw + \int_\mu g(w)\, dw + \int_\nu g(w) dw.$$
4. Prove Cauchy's integral formula for disks.
5. Show that $f'(z)$ can be computed as a line integral on $\gamma$ of an expression that depends on $f(w)$ and not on $f'(w)$. What property of $f'$ does this expression shows?

**Answers**

1. If $|z-c| > r$, the function $w \mapsto f(w)/(w-z)$ is defined and holomorphic in $\Omega \setminus \{z\}$. Let $\rho$ be the minimum between $|z-c|$ and the distance between $c$ and $\mathbb{C} \setminus \Omega$. By construction, the open disk $D(c, \rho)$ is a star-shaped subset of $\Omega \setminus \{z\}$ and it contains the image of $\gamma$. Thus, Cauchy's integral theorem provides
$$\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w-z} dw = 0.$$

2. When $z = c$, we have
$$\begin{aligned} \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w-z} dw &= \frac{1}{i2\pi} \int_0^1 \frac{f(c + re^{i2\pi t})}{c + re^{i2\pi t} - c} (i2\pi r e^{i2\pi t} dt) \\ &= \int_0^1 f(c + re^{i2\pi t}) dt. \end{aligned}$$
By continuity of $f$ at $c$, the limit of this integral when $r \to 0$ is $f(c)$.

3. Assume for the sake of simplicity that $z = c + x$ for some real number $x \in [0, r[$. Let $\alpha = \arccos x/r$; define $\mu$ as the concatenation
$$\begin{array}{rcll} \mu & = & c + [x + i\epsilon \to re^{i\alpha}] & | \\ & & c + re^{i[\alpha \to 2\pi - \alpha]} & | \\ & & c + [re^{-i\alpha} \to x - i\epsilon] & | \\ & & c + x + \epsilon e^{i[-\pi/2 \to -3\pi/2]} & . \end{array}$$
and $\nu$ as the concatenation
$$\begin{array}{rcll} \nu & = & c + [x - i\epsilon \to re^{-i\alpha}] & | \\ & & c + re^{i[-\alpha \to \alpha]} & | \\ & & c + [re^{i\alpha} \to x + i\epsilon] & | \\ & & c + x + \epsilon e^{i[\pi/2 \to -\pi/2]} & . \end{array}$$
Since the closure of $D(c, r)$ is included in $\Omega$, there is a $\rho > r$ such that $D(c, \rho) \subset \Omega$. The image of $\mu$ belongs to the set
$$D(c, \rho) \setminus \{z + t \mid t \geq 0\}$$
while the image of $\nu$ belongs to the set
$$D(c, \rho) \setminus \{z + t \mid t \leq 0\}.$$
Both sets are star-shaped and included in $\Omega$.

   Additionally, for any continuous function $g : \Omega \setminus \{z\} \to \mathbb{C}$

   - the integral of $g$ on $c + [x + i\epsilon \to re^{i\alpha}]$ and its reverse path on one hand, the integral of $g$ on $c + [re^{-i\alpha} \to x - i\epsilon]$ and its reverse path on the other hand are opposite numbers.

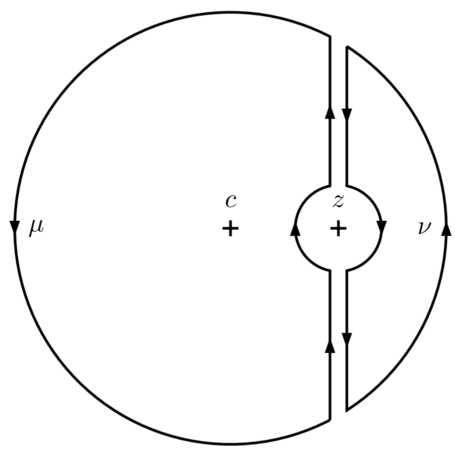


Figure 3: Cauchy's Integral Formula for Disks

- the sum of the integral of $g$ on $c + re^{i[\alpha \to 2\pi - \alpha]}$ and $c + re^{i[-\alpha \to \alpha]}$ is equal to its integral on $\gamma = c + r[\circlearrowleft]$.
- the sum of the integral of $g$ on $c + x + \epsilon e^{i[-\pi/2 \to -3\pi/2]}$ and $c + x + \epsilon e^{i[\pi/2 \to -\pi/2]}$ is equal to the opposite of its integral on $\lambda = c + x + \epsilon[\circlearrowleft]$.

Therefore, the equality

$$\int_\gamma g(w)\, dw = \int_\lambda g(w) dw + \int_\mu g(w)\, dw + \int_\nu g(w) dw.$$

holds.

4. We may apply the result of the previous question to the function $w \mapsto f(w)/(w-z)$. As it is holomorphic on $\Omega \setminus \{z\}$, the star-shaped version of Cauchy's integral theorem provides

$$\int_\mu \frac{f(w)}{w-z} dw = \int_\nu \frac{f(w)}{w-z} dw = 0,$$

hence

$$\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w-z} dw = \frac{1}{i2\pi} \int_\lambda \frac{f(w)}{w-z} dw.$$

We proved in question 2. that the right-hand side of this equation tends to $f(z)$ when $\epsilon \to 0$, thus

$$\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w-z} dw = f(z),$$

which is Cauchy's integral formula for disks.

5. Let $z \in \Omega$. There are some $c \in \Omega$ and $r > 0$ such that $z \in D(c, r)$ and $\overline{D}(z, r) \subset \Omega$; let $\gamma = c + r[\circlearrowleft]$. For any complex number $h$ such that $|z + h - c| < r$, we have by Cauchy's formula

$$\begin{aligned}\frac{f(z+h) - f(z)}{h} &= \frac{1}{i2\pi} \int_\gamma \frac{1}{h} \left( \frac{1}{w - z - h} - \frac{1}{w - z} \right) f(w)\, dw \\ &= \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{(w - z - h)(w - z)}\, dw\end{aligned}$$

To met the condition on $h$, assume that $|h| \leq \epsilon$ where

$$0 < \epsilon < \min_{t \in [0,1]} |z - \gamma(t)|.$$

Write the line integral above as an integral with respect to the real parameter $t \in [0, 1]$; its integrand is dominated by a constant:

$$\forall\, t \in [0, 1], \; \left| \frac{1}{i2\pi} \frac{f(\gamma(t))}{(\gamma(t) - z - h)(\gamma(t) - z)} \gamma'(t) \right| \leq \frac{1}{\epsilon^2} \max_{t \in [0,1]} |f(\gamma(t))|.$$

Thus, Lebesgue's dominated convergence theorem provides the existence of the derivative of $f$ at $z$ as well at its value:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{(w - z)^2}\, dw.$$

Now it's pretty clear that we can iterate the previous argument: consider the right-hand side of the above equations as a function of $z$, build its difference quotient and pass to the limit. The process provides

$$f''(z) = \lim_{h \to 0} \frac{f'(z+h) - f'(z)}{h} = \frac{1}{i2\pi} \int_\gamma \frac{2f(w)}{(w - z)^3}\, dw.$$

The argument is valid for any $z \in \Omega$: the function $f'$ is also holomorphic.

## The Fundamental Theorem of Algebra

### Question

Prove that every non-constant single-variable polynomial with complex coefficients has at least one complex root.

### Answer

Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial with no complex root. The function $f = 1/p$ is defined and holomorphic on $\mathbb{C}$. Additionally, as $|p(z)| \to +\infty$ when $|z| \to +\infty$, the modulus of $f$ is bounded. By Liouville's theorem, $f$ is constant, hence $p$ is constant too.

## Image of Entire Functions

### Question

Show that any non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ has an image which is dense in $\mathbb{C}$:

$$\forall\, w \in \mathbb{C},\ \forall\, \epsilon > 0,\ \exists\, z \in \mathbb{C},\ |f(z) - w| < \epsilon.$$

### Answer

Assume that the image of $f$ is not dense in $\mathbb{C}$: there is a $w \in \mathbb{C}$ and a $\epsilon > 0$ such that for any $z \in \mathbb{C}$, $|f(z) - w| \geq \epsilon$. Now consider the function $z \mapsto 1/(f(z) - w)$; it is defined and holomorphic in $\mathbb{C}$. Additionally,

$$\forall\, z \in \mathbb{C},\ \left|\frac{1}{f(z) - w}\right| \leq \frac{1}{\epsilon}.$$

By Liouville's theorem, this function is constant, hence $f$ is constant too.