

# Cauchy's Integral Theorem – Global Version

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## Exercises

### Cauchy's Converse Integral Theorem

#### Question

Let  $\Omega$  be an open subset of  $\mathbb{C}$ .

Suppose that for every holomorphic function  $f : \Omega \rightarrow \mathbb{C}$ ,

$$\int_{\gamma} f(z) dz = 0$$

for some sequence  $\gamma$  of closed rectifiable paths of  $\Omega$ . What conclusion can we draw? What if the property holds for every sequence  $\gamma$  of closed rectifiable paths of  $\Omega$ ?

#### Answer

For any  $w \in \mathbb{C} \setminus \Omega$ , the function  $f : z \in \Omega \mapsto 1/(z-w)$  is defined and holomorphic, thus

$$\text{ind}(\gamma, w) = \frac{1}{i2\pi} \int_{\gamma} \frac{dz}{z-w} = 0$$

and therefore  $\text{Int } \gamma \subset \Omega$ . Now, suppose that this conclusion holds for any sequence  $\gamma$  of closed rectifiable paths of  $\Omega$ . Since the winding number is locally constant and since for any closed path  $\mu$  of  $\Omega$  and any  $\epsilon > 0$ , there is a closed rectifiable path  $\gamma$  of  $\Omega$  such that

$$\forall t \in [0, 1], |\gamma(t) - \mu(t)| < \epsilon,$$

we also have  $\text{ind}(\mu, w) = \text{ind}(\gamma, w) = 0$ . Therefore  $\text{Int } \mu \subset \Omega$ : the set  $\Omega$  is simply connected.

## Cauchy Transform of Power Functions

### Question

Compute for any  $n \in \mathbb{Z}$  and any  $z \in \mathbb{C}$  such that  $|z| \neq 1$  the line integral

$$\phi(z) = \frac{1}{i2\pi} \int_{[\odot]} \frac{w^n}{w-z} dw.$$

### Answer

For any  $z \in \mathbb{C}$  such that  $|z| \neq 1$ , the function

$$\psi_z : w \mapsto \frac{w^n}{w-z}$$

is defined and holomorphic on  $\Omega = \mathbb{C} \setminus \{z\}$  if  $n \geq 0$ ; it is defined and holomorphic on  $\Omega = \mathbb{C} \setminus \{0, z\}$  if  $n < 0$ . The interior of  $[\odot]$  is the open unit disk.

We now study separately four configurations.

1. Assume that  $n \geq 0$  and  $|z| > 1$ . The interior of  $[\odot]$  is included in  $\Omega$ , hence by Cauchy's integral theorem,  $\phi(z) = 0$ .

Alternatively, Cauchy's formula was also applicable.

2. Assume that  $n \geq 0$  and  $|z| < 1$ . The unique singularity of  $\psi_z$  is  $w = z$ ; it satisfies  $\text{ind}([\odot], z) = 1$ . Let  $\gamma(r) = z + r[\odot]$ ; we have

$$\text{res}(\psi_z, z) = \lim_{r \rightarrow 0} \frac{1}{i2\pi} \int_{\gamma(r)} \frac{w^n}{w-z} dw = \lim_{r \rightarrow 0} \int_0^1 (z + re^{i2\pi t})^n dt = z^n,$$

hence by the residues theorem,  $\phi(z) = z^n$ .

Alternatively, Cauchy's formula was also applicable.

3. Assume that  $n < 0$  and  $|z| > 1$ . We have

$$\phi(z) = \frac{1}{i2\pi} \int_{[\odot]} \frac{1}{w^{|n|}(w-z)} dw.$$

If  $n = -1$ , Cauchy's formula provides the answer:

$$\phi(z) = \frac{1}{0 - z} = -z^{-1}.$$

Otherwise  $n < -1$ , we may use integration by parts (several times):

$$\begin{aligned} \phi(z) &= \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{1}{w^{|n|}(w-z)} dw \\ &= -\frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{-1}{|n|-1} \frac{1}{w^{|n|-1}} \frac{-1}{(w-z)^2} dw \\ &= \dots \\ &= (-1)^{|n|-1} \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{1}{(|n|-1)!} \frac{1}{w} \frac{(|n|-1)!}{(w-z)^{|n|}} dw \\ &= -\frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{1}{w} \frac{1}{(z-w)^{|n|}} dw. \end{aligned}$$

At this point, Cauchy's formula may be used again and we obtain

$$\phi(z) = -z^n.$$

Alternatively, we may perform the change of variable  $w = 1/\xi$  :

$$\begin{aligned} \phi(z) &= \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{w^n}{w-z} dw \\ &= -\frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{\xi^{-n}}{\xi^{-1}-z} \left(-\frac{d\xi}{\xi^2}\right) \\ &= -\frac{1}{z} \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{\xi^{-n-1}}{\xi-z^{-1}} d\xi. \end{aligned}$$

As  $-n-1 \geq 0$  and  $|z^{-1}| < 1$ , we may invoke the result obtained for the first configuration: it provides  $\phi(z) = -z^n$ .

Alternatively, we may perform a partial fraction decomposition of  $w \mapsto 1/(w^{|n|}(w-z))$ . Since

$$1 - \left(\frac{w}{z}\right)^{|n|} = \left(1 - \frac{w}{z}\right) \left(1 + \frac{w}{z} + \dots + \left(\frac{w}{z}\right)^{|n|-1}\right),$$

we have

$$\frac{1}{w-z} = -\frac{1}{z} \left(1 + \frac{w}{z} + \dots + \left(\frac{w}{z}\right)^{|n|-1}\right) + \frac{w^{|n|}/z^{|n|}}{w-z}$$

and therefore

$$\frac{1}{w^{|n|}(w-z)} = -\left(\frac{z}{w^{|n|}} + \frac{1/z^2}{w^{|n|-1}} + \dots + \frac{1/z^{|n|}}{w^{-1}}\right) + \frac{1/z^{|n|}}{w-z}.$$

The integral along  $\gamma$  of  $w \in \mathbb{C} \mapsto 1/w^p$  is zero for  $p > 1$  since this function has a primitive. The integral of  $w \mapsto 1/(w - z)$  is also zero since  $|z| > 1$ . Finally,

$$\phi(z) = \frac{1}{i2\pi} \int_{\gamma} -\frac{1/z^{|n|}}{w^{-1}} dw = -z^n.$$

4. Assume that  $n < 0$  and  $|z| < 1$ . There are two singularities of  $\psi_z$  in the interior of  $[\odot]$ ,  $w = 0$  and  $w = z$ , unless of course if  $z = 0$ .

If  $z = 0$ , we have

$$\phi(z) = \frac{1}{i2\pi} \int_{[\odot]} w^{n-1} dw = 0$$

because  $w \in \mathbb{C}^* \mapsto w^n/n$  is a primitive of  $w \in \mathbb{C}^* \mapsto w^{n-1}$ .

We now assume that  $z \neq 0$ . The residue associated to  $w = z$  can be computed directly with Cauchy's formula; with  $\gamma(r) = z + r[\odot]$ , we have

$$\text{res}(\psi_z, z) = \lim_{r \rightarrow 0} \frac{1}{i2\pi} \int_{\gamma(r)} \frac{w^n}{(w - z)} dw = z^n.$$

On the other hand, using computations similar to those of the previous question, we can derive

$$\text{res}(\psi_z, 0) = \lim_{r \rightarrow 0} \frac{1}{i2\pi} \int_{r[\odot]} \frac{w^n}{(w - z)} dw = -z^n.$$

Consequently,  $\phi(z) = 0$ .

In the case  $z \neq 0$ , we may perform again the change of variable  $w = 1/\xi$  that provides

$$\phi(z) = -\frac{1}{z} \frac{1}{i2\pi} \int_{[\odot]} \frac{\xi^{-n-1}}{\xi - z^{-1}} d\xi.$$

As  $-n - 1 \geq 0$  and  $|z^{-1}| > 1$ , we may invoke the result obtained for the first configuration: it yields  $\phi(z) = 0$ .

There is yet another method: we can notice that for  $r > 1$ , the interior of the path sequence  $(r[\odot], [\odot]^\leftarrow)$ , which is the annulus  $\{z \in \mathbb{C} \mid 1 < |z| < r\}$ , is included in  $\Omega$ . Cauchy's integral theorem provides

$$\forall r > 1, \phi(z) = \frac{1}{i2\pi} \int_{r[\odot]} \frac{w^n}{w - z} dw.$$

and the M-L estimation lemma

$$\forall r > 1, |\phi(z)| \leq \frac{1}{r^{|n|-1}(r - |z|)}.$$

The limit of the right-hand side when  $r \rightarrow +\infty$  yields  $\phi(z) = 0$ .

Finally, we may use again the partial fraction decomposition of  $w \mapsto 1/(w^{|n|}(w-z))$ :

$$\frac{1}{w^{|n|}(w-z)} = -\left(\frac{z}{w^{|n|}} + \frac{1/z^2}{w^{|n|-1}} + \cdots + \frac{1/z^{|n|}}{w^{-1}}\right) + \frac{1/z^{|n|}}{w-z}.$$

The integral along  $\gamma$  of  $w \in \mathbb{C} \mapsto 1/w^p$  is zero for  $p > 1$  since this function has a primitive. Therefore

$$\phi(z) = \frac{1}{i2\pi} \int_{\gamma} -\frac{1/z^{|n|}}{w^{-1}} dw + \frac{1}{i2\pi} \int_{\gamma} \frac{1/z^{|n|}}{w-z} dw = 0.$$