Analytic Functions

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Definition – Analytic Function. A function $f : A \subset \mathbb{C} \to \mathbb{C}$ is *analytic* if it is locally the sum of a (convergent) power series: for any $c \in A$, there is a $r > 0$ and a sequence of complex numbers a_n such that

$$\forall \, z \in A \cap D(c, r), \; f(z) = \sum_{n=0}^{+\infty} a_n (z - c)^n.$$

The characterization of analytic functions defined on open sets is simple: we know that the sum of every power series is holomorphic in its open disk of convergence and conversely that every holomorphic function defined on an open set is locally the sum of a power series. Hence, we have the:

Theorem – Analyticity & Holomorphicity in Open Sets. A function defined in an open set is analytic if and only if it is holomorphic.

The definition of analytic function is not limited to open sets; we may also extend the definition of holomorphic function to sets that may not be open:

Definition – Holomorphic Function (Non-Open Sets). A function $f : A \subset \mathbb{C} \to \mathbb{C}$ is *holomorphic* if there is an open subset Ω of $\mathbb{C}$ such that $A \subset \Omega$ and a holomorphic function $g : \Omega \to \mathbb{C}$ that extends f.

This definition is appropriate because – at least on "well-behaved" sets – the classes of analytic and holomorphic functions are identical, as they are on open sets. For example, convex sets are well-behaved:

Theorem – Analyticity & Holomorphicity in Convex Sets. A function $f : A \subset \mathbb{C} \to \mathbb{C}$ defined in a convex set A is analytic if and only if it is holomorphic.

Proof. It is clear that such restrictions of holomorphic functions are analytic (the convexity assumption is not needed here).

Conversely, let A be a convex subset of $\mathbb{C}$ and let $f : A \to \mathbb{C}$ be an analytic function. There is a positive function $a \in A \mapsto r_a$ and a family $a \in A \mapsto f_a$ of holomorphic functions defined on $D(a, r_a)$ such that f and f_a are equal on $A \cap D(a, r_a)$. Let ϕ be the complex-valued function defined on the union Ω of all disks $D(a, r_a)$ by

$$\phi(z) = f_a(z) \;\text{ if }\; a \in A \;\text{ and }\; z \in D(a, r_a).$$

This definition is non-ambiguous: if $a \in A$, $b \in A$ and z belongs to $D(a, r_a)$ and $D(b, r_b)$, then by convexity of A, the domain of f contains the set $L = [a, b] \cap D(a, r_a) \cap D(b, r_b)$. By the isolated zeros theorem, the functions f_a and f_b, equal on L, are also equal on $D(a, r_a) \cap D(b, r_b)$, therefore $f_a(z) = f_b(z)$. The function ϕ is by construction an extension of f which is holomorphic. ■

Example – Closed Unit Disk. If the function f is analytic on the closed unit disk

$$\overline{D}(0, 1) = \{z \in \mathbb{C} \mid |z| \leq 1\},$$

there is a holomorphic extension of f on an open superset Ω of the closed unit disk. This function may be restricted to an open disk $D(0, r)$ for some radius $r > 1$ and expanded into a Taylor series: there are complex coefficients a_n such that $\sum_{n=0}^{+\infty} a_n z^n$ is convergent in $D(0, r)$ and

$$\forall\, z \in \overline{D}(0, 1),\; f(z) = \sum_{n=0}^{+\infty} a_n z^n.$$

Conversely it is plain that such a function is holomorphic – and thus analytic – in the closed unit disk.

Definition – Projection. A *projection* onto a non-empty closed subset A of $\mathbb{C}$ of a point $z \in \mathbb{C}$ is a minimizer of the distance between z and A. In other words, the set of all projections of z onto A is

$$\Pi(z) = \{a \in A \mid |z - a| = \inf_{w \in A} |z - w|\}.$$

Theorem – Analyticity & Holomorphicity in Regular Compact Sets. Let A be a compact subset of the complex plane with no isolated point and such that the projection onto A is unique in some neighbourhood of A. A function $f : A \to \mathbb{C}$ is analytic if and only if it is holomorphic.

Proof. It is plain that restrictions to A of holomorphic functions defined in a open neighbourhood of A are analytic on A : holomorphic functions are analytic.

Conversely, assume that $f : A \to \mathbb{C}$ is analytic. Let r_z be a collection of positive radii indexed by $z \in A$ and $f_z : D(z, r_z) \to \mathbb{C}$ the corresponding collection of holomorphic functions such that

$$\forall\, z \in A,\ \forall\, w \in A \cap D(z, r_z),\ f_z(w) = f(w).$$

There is another collection with the same property, but associated with a constant and positive radius r. It is easy to build this collection – by a simple restriction of the f_z – if we can first extend the functions of the original collection to make sure that $z \in A \mapsto r_z$ has a positive lower bound. To do so, we may consider for any $z \in A$ the set of holomorphic functions defined on open disks centered on z with the same values as f on the points that also belong to A. This set is not empty, as it contains f_z. Additionally, if two functions belong to this set, as z is not isolated in A, they are equal on the intersection of their domain of definition. Hence, we may select as a new f_z the function in the set whose domain of definition has the largest radius. Now, if we assume that the infimum of the r_z is 0, there is a sequence z_n in A such that $r_{z_n} \to 0$ and thus a subsequence w_n that converges to some $z \in A$. But for any $w \in D(z, r_z)$, the radius of the largest disk at w satisfies

$$r_w + |w - z| \geq r_z,$$

thus we have a contradiction eventually when $|w_n - z| < r_z/2$ and $r_{w_n} < r_z/2$.

Let V be an open neighbourhood of A where the projection onto A is unique. We may assume that r is smaller than the distance between A and $\mathbb{C} \setminus V$. For any $z \in \Omega = A + D(0, r)$, denote $\pi(z)$ the projection of z onto A and define $g(z) = f_{\pi(z)}(z)$; the function $g : \Omega \mapsto \mathbb{C}$ obviously extends f. Additionally, it is holomorphic, because it satisfies

$$\forall\, z \in \Omega,\ \exists\, \epsilon > 0,\ \forall\, w \in D(z, \epsilon),\ g(w) = f_{\pi(z)}(w).$$

Indeed, if $|w - z| < \epsilon = r - |z - \pi(z)|$, then

$$|w - \pi(z)| \leq |w - z| + |z - \pi(z)| < r,$$

so w that belongs to $D(\pi(w), r)$ by construction also belongs to $D(\pi(z), r)$. By construction, the functions $f_{\pi(z)}$ and $f_{\pi(w)}$ have the same values on every point of A that belongs to both of their domains of definition. Naturally, $\pi(z)$ belongs to the domain of $f_{\pi(z)}$; since

$$|\pi(w) - \pi(z)| \leq |w - z| < r$$

it also belongs to the domain of $f_{\pi(w)}$. As none of the points of A is isolated, by the isolated zeros theorem, the functions $f_{\pi(w)}$ and $f_{\pi(z)}$ are identical on $D(\pi(z), r) \cap D(\pi(w), r)$, which leads to $g(w) = f_{\pi(w)}(w) = f_{\pi(z)}(w)$. ■

Example – Unit Circle. The unit circle

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$$

is not convex, but it is compact and has no isolated point. Additionally, unless $z = 0$, the point $z/|z|$ is the unique projection of z onto $\mathbb{C}$; the set $\mathbb{C}^*$ is a neighbourhood of $\mathbb{U}$. Thus, the classes of analytic and holomorphic functions defined on $\mathbb{U}$ are identical. Any holomorphic extension of a holomorphic function defined on $\mathbb{U}$ has a restriction to $A(0, r, 1/r)$ for some radius $r \in \,]0, 1[$, thus a function f is analytic on $\mathbb{U}$ if and only if there are some coefficients a_n such that $\sum_{n=-\infty}^{+\infty} a_n z^n$ is convergent on some annulus $A(0, r, 1/r)$ and

$$\forall\, z \in \mathbb{U},\ f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n.$$

Real Analytic Functions

Definition – Interval. In the sequel, an interval may be open, closed or half-open, bounded or unbounded; in other words, it is a synonym for "convex subset of the real line".

Definition – Real Analytic Function. A real-valued function defined on an interval of the real line which is analytic is *real analytic*.

The characterization of analytic functions on convex sets provides:

Theorem – Characterization of Real Analytic Functions. A real-valued function defined on an interval of the real line is real analytic if and only if it has a holomorphic extension on some open neighbourhood of its domain of definition in the complex plane.

We also have:

Theorem – (Real) Taylor Series Expansion. A function $f : I \to \mathbb{R}$ defined on an interval I of $\mathbb{R}$ is real analytic if and only it is smooth – the n-th order (real) derivative defined by $f^{(0)} = f$ and

$$f^{(n+1)}(x) = \lim_{y \in I \to x} \frac{f^{(n)}(y) - f^{(n)}(x)}{y - x}$$

exists at every order – and for any $a \in I$, there is a $r > 0$ such that

$$\forall\, x \in I \cap \,]a - r, a + r[\,,\ f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Proof. Assume that the function f is smooth and that for any a in I, it is equal to the sum of its (convergent) Taylor expansion on $]a - r, a + r[$ for some $r > 0$. Then, the power series

$$z \mapsto \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(z - a)^n$$

is convergent on $D(a, r)$ and its restriction to

$$I \cap D(a, r) = I \cap]a - r, a + r[$$

is equal to f. Hence, f is real analytic.

Conversely, if f is real analytic, let g be one of its holomorphic extensions. At every x in I,

$$f'(x) = \lim_{y \in I \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \in I \to x} \frac{g(y) - g(x)}{y - x} = g'(x);$$

the real derivative of f exists and is equal to the complex derivative of g, hence g' is a holomorphic extension of f'. By induction, at every order n, $f^{(n)}$ exists and accepts $g^{(n)}$ as a holomorphic extension. Consequently, the local expansion of g as a Taylor series provides a local expansion of f as a (real) Taylor series. ■

Analytic Continuation

The term "analytic continuation" refers to two related concepts; the first one is straightforward: "continuation" is simply used as a synonym of "function extension".

Definition – Analytic Continuation to a Set. Let Ω be an non-empty open subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function. An *analytic continuation* of f on a superset Σ of Ω which is open and connected is a holomorphic function $g : \Sigma \to \mathbb{C}$ such that the restriction of g to Ω is f.

We define the continuation for open and connected sets only so that:

Theorem – Uniqueness of Analytic Continuation to Sets. There is at most one analytic continuation of an analytic function to a given set.

Proof. It is a consequence of the isolated zeros theorem. Indeed, if there are two holomorphic extensions, they are identical on Ω which is open and non-empty. Every point of Ω is a limit point of the zeros of their difference; this difference is holomorphic and defined on a connected set, thus it is identically zero and the two holomorphic extensions are equal. ■

The second concept of continuation applies to holomorphic functions along paths. Although the concept may be defined for arbitrary paths, for the sake of simplicity we will only consider the case of regular analytic paths.

Definition – Regular Analytic Path. A path $\gamma : [0, 1] \mapsto \mathbb{C}$ is *analytic* if it is an analytic function. It is *regular* if γ' has no zero on $[0, 1]$.

Definition – Analytic Continuation Along a Path. Let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function and $\gamma : [0, 1] \mapsto \mathbb{C}$ be a regular analytic path such that $\gamma(0) \in \Omega$. An *analytic continuation* of f along γ is an analytic function $\phi : [0,1] \mapsto \mathbb{C}$ such that for some $0 < \epsilon \leq 1$,

$$\forall\, t \in [0, \epsilon[\,,\ \gamma(t) \in \Omega \ \text{ and } \ \phi(t) = f(\gamma(t)).$$

Example – Analytic Continuations of the Logarithm. The principal value of the logarithm

$$\log : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$$

is an analytic continuation of the function $f : D(1,1) \mapsto \mathbb{C}$ defined by

$$f(z) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} (z-1)^n.$$

On the other hand, the function

$$\phi : t \in [0,1] \mapsto i2\pi t$$

is an analytic continuation of log along the regular analytic path

$$[\circlearrowleft] : t \in [0,1] \mapsto e^{i2\pi t}.$$

Indeed, the function ϕ is analytic – the holomorphic function $z \in \mathbb{C} \mapsto i2\pi z$ is an extension of it – and

$$\forall\, t \in [0, 1/2[\,,\ [\circlearrowleft](t) = e^{i2\pi t} \in \mathbb{C} \setminus \mathbb{R}_- \ \text{ and } \ \phi(t) = \log e^{i2\pi t} = \log([\circlearrowleft](t)).$$

Like analytic continuations to sets, analytic continuations along paths are also unique whenever they exist:

Theorem – Uniqueness of Analytic Continuation along Paths. There is at most one analytic continuation of an analytic function along a given regular analytic path.

Proof. If $\phi_1 : [0,1] \to \mathbb{C}$ and $\phi_2 : [0,1] \to \mathbb{C}$ are two analytic continuations of the same holomorphic function along the same path, they have holomorphic extensions – still denoted ϕ_1 and ϕ_2 – defined on some shared open tubular neighbourhood of $[0,1]$

$$\{z \in \mathbb{C} \mid d(z, [0,1]) < \epsilon\},$$

which is connected. The point 0 is a limit point of the zeros of the holomorphic function $\phi_1 - \phi_2$. By the isolated zeros theorem, ϕ_1 and ϕ_2 are identical. ■

Remark – Comparison of Analytic Continuations Concepts. Continuation along paths is a more flexible tool than continuation to sets: if $\phi : \Sigma \mapsto \mathbb{C}$ is an analytic continuation of $f : \Omega \mapsto \mathbb{C}$ to the open connected set Σ and γ is a regular analytic path of Σ whose initial point is in Ω, then $\phi \circ \gamma$ is always an analytic continuation of f along γ. However, on the other hand, there may exist an analytic continuation of f along a path γ but no analytic continuation $\phi : \Sigma \mapsto \mathbb{C}$ such that $\gamma([0,1]) \subset \Sigma$ (refer e.g. to the logarithm example).

Remark – Values of Analytic Continuations. The analytic continuation of a function f to an open connected set Σ provides a unique "natural" value of the function f associated to a point $z \in \Sigma$ that may be outside of the original function domain. An analytic continuation along a path ϕ also provides a new value associated to the terminal point $z = \gamma(1)$: the terminal value $\phi(1)$ of the continuation ϕ. However, several analytic continuations of f along paths with the same initial and terminal points may have different terminal values; thus, analytic continuations along paths may define multi-valued functions.

Example – Values of the Logarithm. The analytic continuation of log along the path $[\circlearrowleft]$ – whose initial and terminal point is $z = 1$ – is the function $t \in [0,1] \mapsto i2\pi t$; its terminal value is $i2\pi$. On the other hand, the path $\gamma = 2 - [\circlearrowleft]$ has the same endpoints as $[\circlearrowleft]$, but its image is included in $\mathbb{C} \setminus \mathbb{R}_-$. Thus, the function $t \in [0,1] \mapsto \log \gamma(t)$ is the analytic continuation of log along γ and its terminal value is $\log 1 = 0$. Both paths have the same initial and terminal points but the terminal values of the corresponding continuations are different.

Actually, an analytic continuation along a path carries more information than a simple value from the initial point to the terminal point of the path: it defines a new holomorphic function in the some open neighbourhood of the terminal point. To formalize this, we introduce a new definition.

Definition – Germ of Holomorphic Function. Two holomorphic functions $f : \Omega \mapsto \mathbb{C}$ and $g : \Sigma \to \mathbb{C}$ *define the same germ* at $z \in \Omega \cap \Sigma$ if the functions are identical in some neighbourghood of z. We denote this relation

$$f \sim_z g$$

A *germ of holomorphic function* at z is an equivalence class for this relation between holomorphic functions:

$$[f]_z = \{g \mid f \sim_z g\}.$$

Theorem – Analytic Continuation of Germs. The existence and definition of an analytic continuation of a holomorphic function $f : \Omega \to \mathbb{C}$ along the regular analytic path γ depends only on the germ of f at $\gamma(0)$.

Proof. Assume that $g : \Sigma \to \mathbb{C}$ defines the same germ as f at $\gamma(0)$; let $\Gamma \subset \Omega \cap \Sigma$ be some open set that contains $\gamma(0)$ and where both functions are identical. Let $\epsilon > 0$ be such that

$$\forall\, t \in [0, \epsilon[\,,\ \gamma(t) \in \Omega \ \text{ and } \ \phi(t) = f(\gamma(t)).$$

and η be some positive number, smaller than ϵ, such that

$$\forall\, t \in [0, \eta[\,,\ \gamma(t) \in \Gamma.$$

Then, by construction of Γ, it is plain that

$$\forall\, t \in [0, \eta[\,,\ \gamma(t) \in \Sigma \ \text{ and } \ \phi(t) = g(\gamma(t)),$$

thus ϕ is also a continuation of g. ■

There is a converse statement that holds true: a candidate continuation function does determine uniquely the germ that it continues. We encapsulate that result in a definition:

Definition – Initial Germ. If $\gamma : [0,1] \mapsto \mathbb{C}$ is a regular analytic path and $\phi : [0,1] \mapsto \mathbb{C}$ is an analytic function, there is a unique germ of holomorphic function at $\gamma(0)$ of which ϕ is an analytic continuation: the *initial germ* of the continuation.

Proof. A function $f : \Omega \mapsto$ is an element of the germ of which ϕ is an analytic continuation along γ if there is a $\epsilon \in \,]0, 1]$ such that

$$\forall\, t \in [0, \epsilon[\,,\ \gamma(t) \in \Omega \ \text{ and } \ \phi(t) = f(\gamma(t)).$$

The uniqueness of such a germ is a consequence of the isolated zeros theorem; let's prove its existence. The functions ϕ and γ are analytic, therefore there are holomorphic functions, still denoted by the symbols ϕ and γ, that extend the original functions to some non-empty open tubular neighbourhood Σ of $[0, 1]$:

$$\Sigma = \{z \in \mathbb{C} \mid d(z, [0, 1]) < r\}.$$

Since the path γ is regular and analytic, the function $\gamma : D(0, r) \mapsto \mathbb{C}$ is continuously real-differentiable and its real-differential $d\gamma$ is invertible on some non-empty neighbourhood of 0. By the inverse function theorem, there is an open neighbourhood U of 0 included in $D(0, r)$ and an open neighbourhood V of $\gamma(0)$ such that the restriction $\gamma|_U$ of γ to U is a C^1-diffeomorphism. Its inverse satisfies $d(\gamma|_U^{-1})_{\gamma(z)} = (d\gamma_z)^{-1}$; this differential is complex-linear, therefore $\gamma|_U^{-1}$ is holomorphic. Thus we may define a holomorphic function f on V by

$$\forall\, w \in V,\ f(w) = \phi(\gamma|_U^{-1}(w))$$

Now let $\epsilon \in \,]0, 1]$ be such that $[0, \epsilon[\, \subset U$. For any value of $t \in [0, \epsilon[\,$, the point $w = \gamma(t)$ belongs to V and $f(\gamma(t)) = \phi(t)$. ■

Definition – Terminal Germ. Let γ be a regular analytic path and let $\phi : [0,1] \mapsto \mathbb{C}$ be an analytic continuation of a germ of holomorphic function at $\gamma(0)$. The *terminal germ* of the continuation ϕ is the initial germ of the continuation $\phi^{\leftarrow} : t \in [0, 1] \mapsto \phi(1-t)$ along $\gamma^{\leftarrow}$; in other words, an holomorphic function $f : \Omega \mapsto \mathbb{C}$ belongs to the terminal germ if and only if, there is a $\epsilon \in [0, 1]$ such that

$$\forall\, t \in \,]1-\epsilon, 1]\,,\ \gamma(t) \in \Omega \ \text{ and } \ \phi(t) = f(\gamma(t)).$$

Theorem – Monodromy Theorem (Local). Let Ω be an open star-shaped subset of $\mathbb{C}$ that contains the non-empty open disk $D(z_0, r)$ and let $f : D(z_0, r) \to$

$\mathbb{C}$ be a holomorphic function. If for any regular analytic path γ of Ω whose initial point is z_0 an analytic continuation of f along γ exists, then there is an analytic continuation of f on Ω.

Proof. We will assume in the sequel that z_0 is a center c of the star-shaped set Ω: if this is not true, consider a center c of Ω and the germ f_c of the holomorphic function defined at c by the analytic continuation of f along $[z_0, c]$. A function defined on Ω is an analytic continuation of f_c if and only if it is an analytic continuation of the original germ f.

Let $z \in \Omega$ and let ϕ_z be the analytic continuation of f along $[c \to z]$. By construction, there is a $\epsilon \in]0,1]$ such that $\phi_z(t) = f((1-t)c + tz)$ for any $0 \leq t < \epsilon$. If we still denote ϕ_z the holomorphic extension of ϕ_z to some open neighbourghood $\{w \in \mathbb{C} \mid d(w,[0,1]) < r\}$ of $[0,1]$, then for any complex number w in some non-empty open disk centered at 0, we have

$$\phi_z(w) = f((1-w)c + wz).$$

Now, for any $z \in \Omega \setminus D(z_0, r)$ and any complex number h such that $|h| < r|z-c|$, the function

$$\psi_{z,h} : t \in [0,1] \mapsto \phi_z(w) \;\; \text{with} \;\; w = t\frac{z+h-c}{z-c}$$

is defined, analytic and as w satisfies

$$(1-t)c + t(z+h) = (1-w)c + wz,$$

for small values of t, we have

$$\psi_{z,h}(t) = \phi_z(w) = f((1-w)c + wz) = f((1-t)c + t(z+h)),$$

therefore $\psi_{z,h}$ is the analytic continuation of f_c along $[c \to z+h]$:

$$\psi_{z,h} = \phi_{z+h}.$$

This leads for small values of h to the equality

$$\phi_{z+h}(1) = \psi_{z,h}(1) = \phi_z\left(\frac{z+h-c}{z-c}\right).$$

The function $z \in \Omega \mapsto \phi_z(1)$ is an extension of f to Ω; by the above equality it is also holomorphic, thus it is the analytic extension of f to Ω. ■