Quantization

Digital Audio Coding

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Scalar Quantizer

$$[\,\cdot\,] : \mathbb{R} \to \mathbb{R}$$

countable range: $|\{[x],\ x \in \mathbb{R}\}| \leq |\mathbb{N}|$

idempotent mapping: $\forall\, x \in \mathbb{R},\ [[x]] = [x]$

Example: rounding functions $\mathbb{R} \to \mathbb{Z} \subset \mathbb{R}$

$$\lfloor \cdot \rfloor,\ \lceil \cdot \rceil,\ [\ \cdot\ ],$$

NumPy: floor, ceil, round_

Forward/Inverse Quantizers

forward quantizer: mapping $i[\,\cdot\,] : \mathbb{R} \to \mathbb{Z}$

such that $i[x] = i[y]$ iff $[x] = [y]$

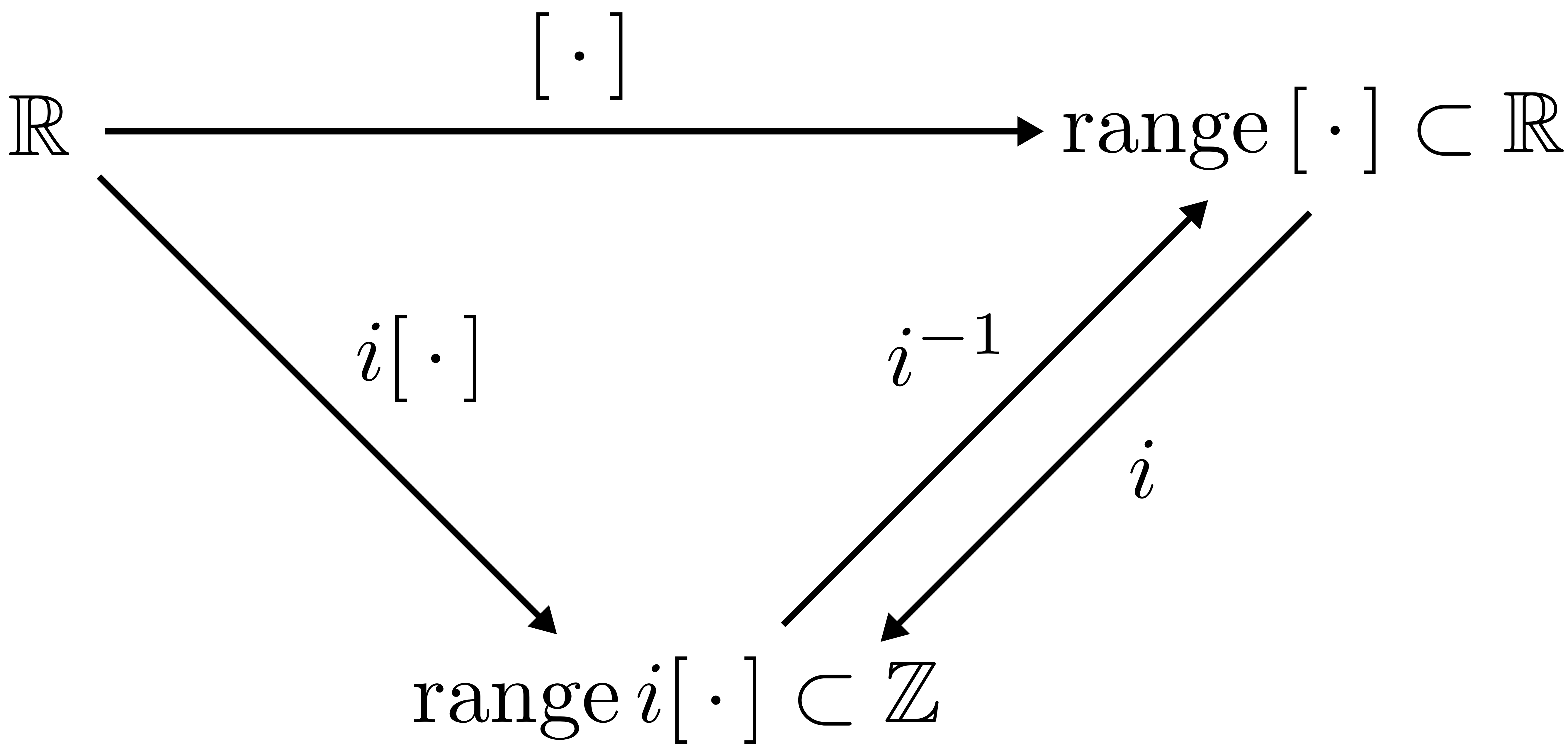
$$i[\,\cdot\,] = i \circ [\,\cdot\,] \;\; \text{where} \;\; i : \text{range}\,[\,\cdot\,] \to \mathbb{Z} \text{ is into}$$

inverse quantizer: $i^{-1} : \text{range}\, i \subset \mathbb{Z} \to \mathbb{R}$

i^{-1} is a left inverse of i.

$$\forall\, x \in \mathbb{R},\;\; (i^{-1} \circ i)[x] = [x]$$

Forward/Inverse Quantizers

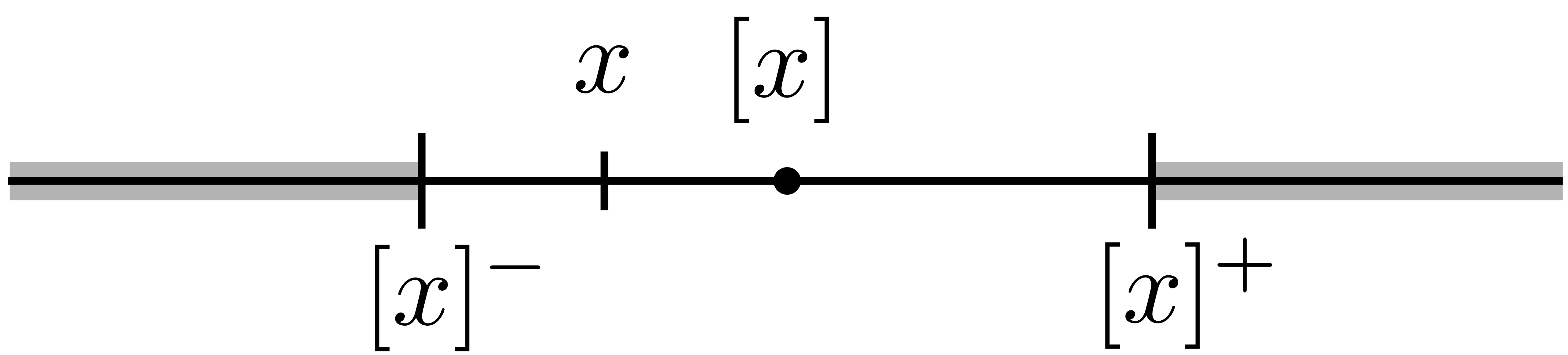


Quantization Step

Assume that for every $x \in \mathbb{R}$,

$$V_x = \{y \in \mathbb{R},\ [y] = [x]\}$$

is an interval.



We define the **decision values:**

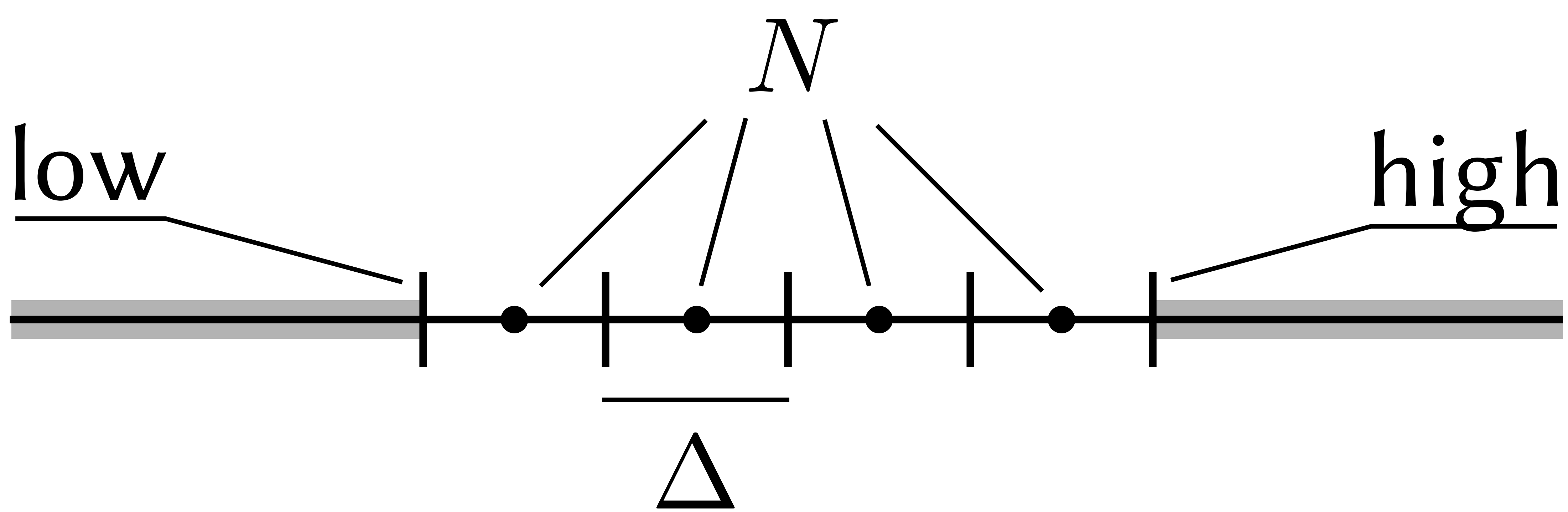
$$[x]^- = \inf V_x \quad \text{and} \quad [x]^+ = \sup V_x$$

and the **quantization step:**

$$\Delta(x) = [x]^+ - [x]^-$$

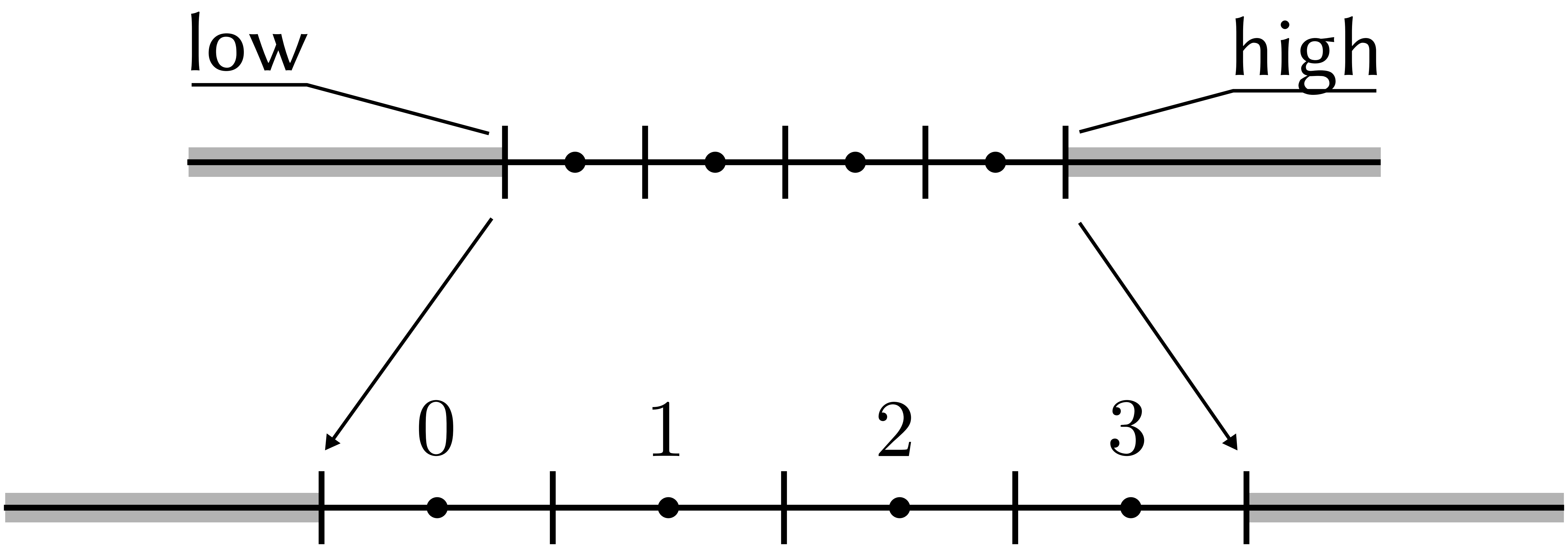
Uniform Quantizer

```
class Uniform(Quantizer):
    def __init__(self, low=0.0, high=1.0, N=2**8):
        self.low, self.high = float(low), float(high)
        self.N = N
        self.delta = (self.high - self.low) / self.N
```

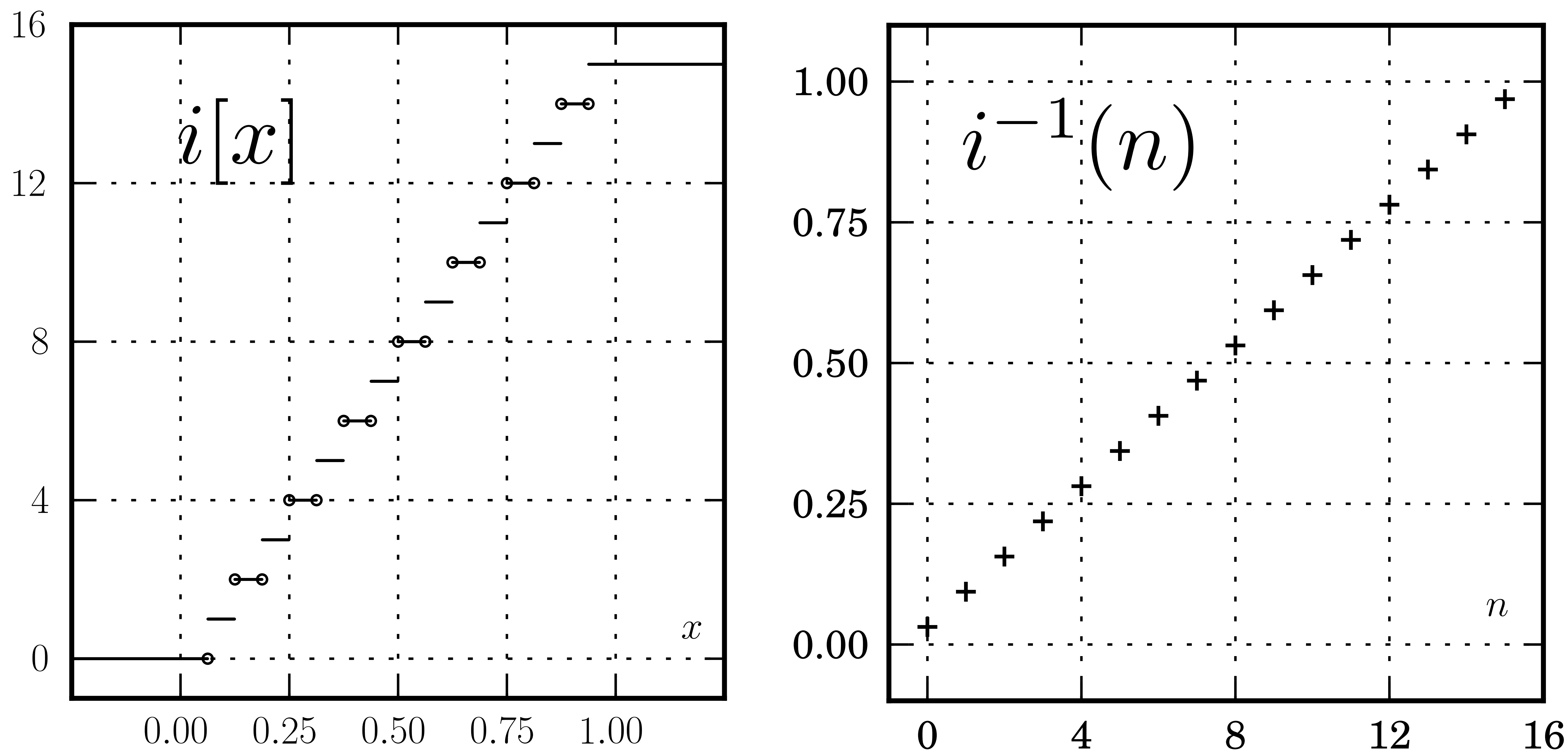


Uniform Quantizer

```
class Uniform(Quantizer):     (continued)
  def encode(self, data):
    data = clip(self.data, self.low, self.high)
    flints = round_((self.data - self.low) / self.delta - 0.5)
    return array(flints, dtype=long)
  def decode(self, i):
    return self.low + (i + 0.5) * self.delta
```



Uniform Quantizer



Uniform(low=0.0, high=1.0, N=2**4)

Uniform Quantizer

```
>>> uniform = Uniform(low=-1.0, high=1.0, N=2**4)
>>> uniform(1.0)
0.9375
>>> uniform(0.0)
0.0625
```

0.0625 —— mid-rise

```
>>> uniform = Uniform(low=-1.0, high=1.0, N=2**4-1)
>>> uniform(1.0)
0.9333333333333335
>>> uniform(0.0)
0.0
```

0.0 —— mid-tread

Random Variables

$$X \in \mathbb{R}, \text{ density } p.$$

$$P([X] = [x]) = P_X\{y \in \mathbb{R},\ [y] = [x]\}$$
$$= \int_{[x]^-}^{[x]^+} p(y)\, dy$$

High Resolution assumption:

$$\approx p(x) \times \Delta(x)$$

Optimal Quantizer

Criteria: Entropy

The **entropy** of $[X]$ is maximal when all events

$$[X] = [x],\ x \in \mathbb{R}$$

are equally likely, that is when:

$$\Delta(x) \propto \frac{1}{p(x)}$$

Nonlinear Quantizers

Select a **characteristic function** f and define

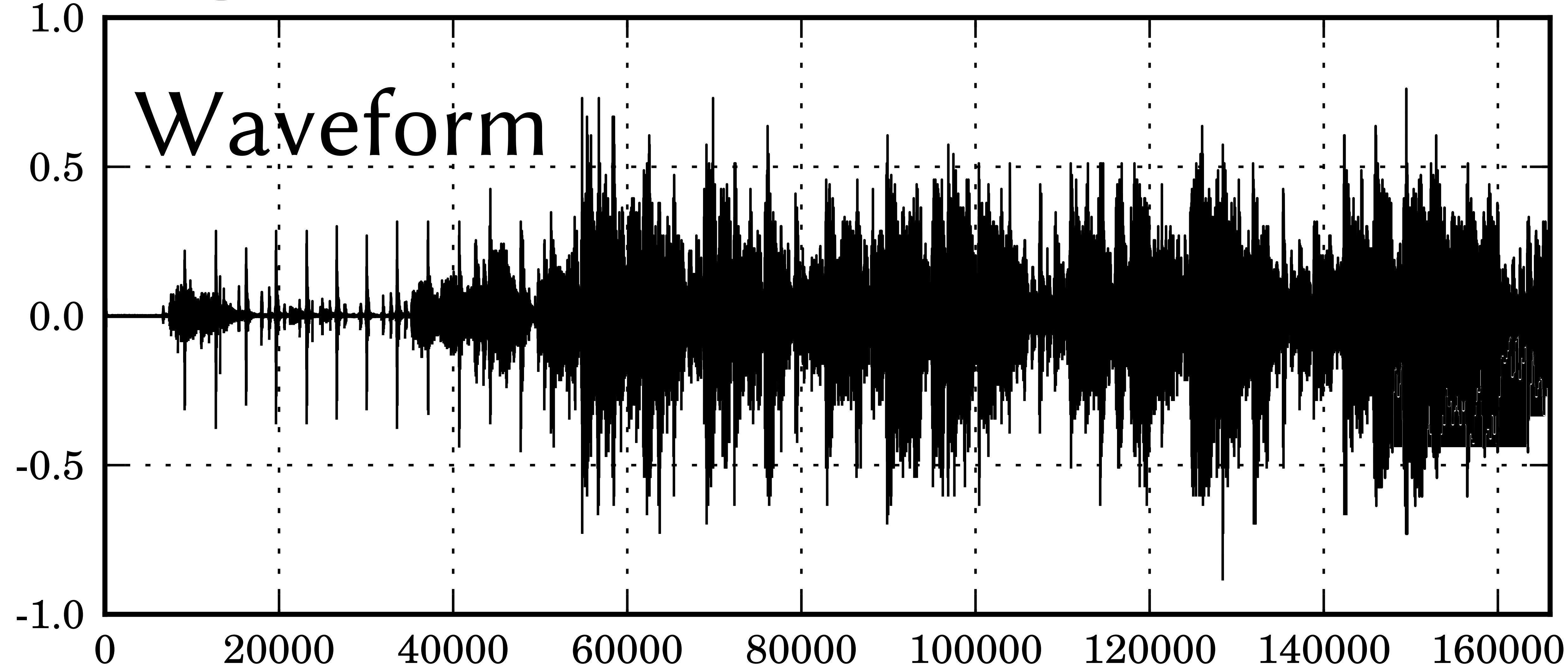
$$[\,\cdot\,]_f = f^{-1} \circ [\,\cdot\,] \circ f$$

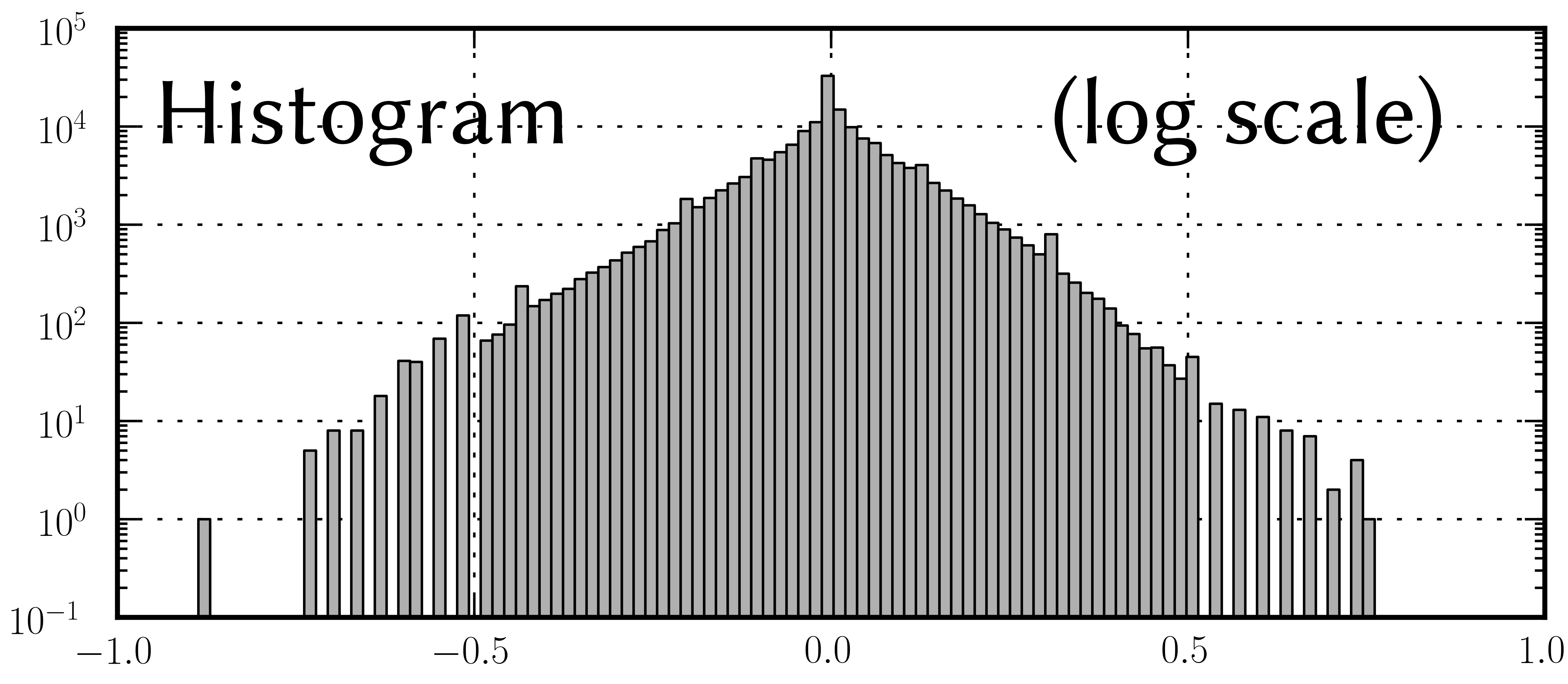
where is $[\,\cdot\,]$ a uniform quantizer.

Under the high resolution assumption:

$$\Delta_f(x) \propto \frac{1}{f'(x)} \qquad \left(\Delta_f(x) = \frac{\Delta_{[\,\cdot\,]}}{f'(x)}\right)$$

"legende.au" (NeXT)





Nonlinear Quantizer

Maximal Entropy

If we model the distribution of values with:

$$p(x) \propto \exp(-a|x|)$$

the optimal quantizer satisfies:

$\Delta(x) \propto \exp(a|x|)$ and $f'(x) \propto \exp(-a|x|)$

Setting $f(0) = 0$ and $f'(0) = a$ yields:

$$f(x) = \operatorname{sgn}(x)(1 - e^{-a|x|})$$

$$f^{-1}(x) = -\frac{\operatorname{sgn}(x)}{a} \log(1 - |x|)$$

Nonlinear Quantizer

Setting $f(0) = 0$ and $f'(0) = a$ yields:

$$f(\mathbb{R}) = (-1, 1)$$

Given implementations **f** and **f_inv** of the characteristic function and of its inverse, we define the optimal 8-bit quantizer with:

```
uniform = Uniform(low=-1.0, high=1.0, N=2**8-1)
quantizer = NonLinear(f, f_inv, uniform)
```

Noise and SNR

Given a random value X and a quantizer $[\,\cdot\,]$, the quantizer **noise** B is defined by:

$$[X] = X + B$$

and the **signal-to-noise ratio (SNR)** by:

$$\mathrm{SNR}^2 = \frac{\mathbb{E}\, X^2}{\mathbb{E}\, B^2}$$

or in **decibels** by:

$$\mathrm{SNR}\,[\mathrm{dB}] = 10 \log_{10} \mathrm{SNR}^2$$

SNR: Number of Bits

Consider a n-bit nonlinear quantizer with

$$f([-1,1]) = [-1,1]$$

The high-resolution assumption yields

$$\mathbb{E}\, B^2 \simeq \frac{1}{12}\mathbb{E}\,\Delta(X)^2 \quad \text{and} \quad \Delta(x) = \frac{2^{-n+1}}{f'(x)}$$

and as a consequence

$$\mathrm{SNR}\,[\mathrm{dB}] \approx 6.0 \times n + c(f)$$

Optimal Quantizer

Criteria: SNR

The best characteristic function is solution of:

$$\min_{f'} \int_{-1}^{1} \frac{1}{f'(x)^2} p(x)\, dx$$

$$\text{subject to } f(1) - f(-1) = 2$$

Solution: $$f'(x) \propto p(x)^{1/3}$$

(Reminder: $f'(x) \propto p(x)$ optimal for the entropy.)

Logarithmic Quantizers

Consider the probability distribution:

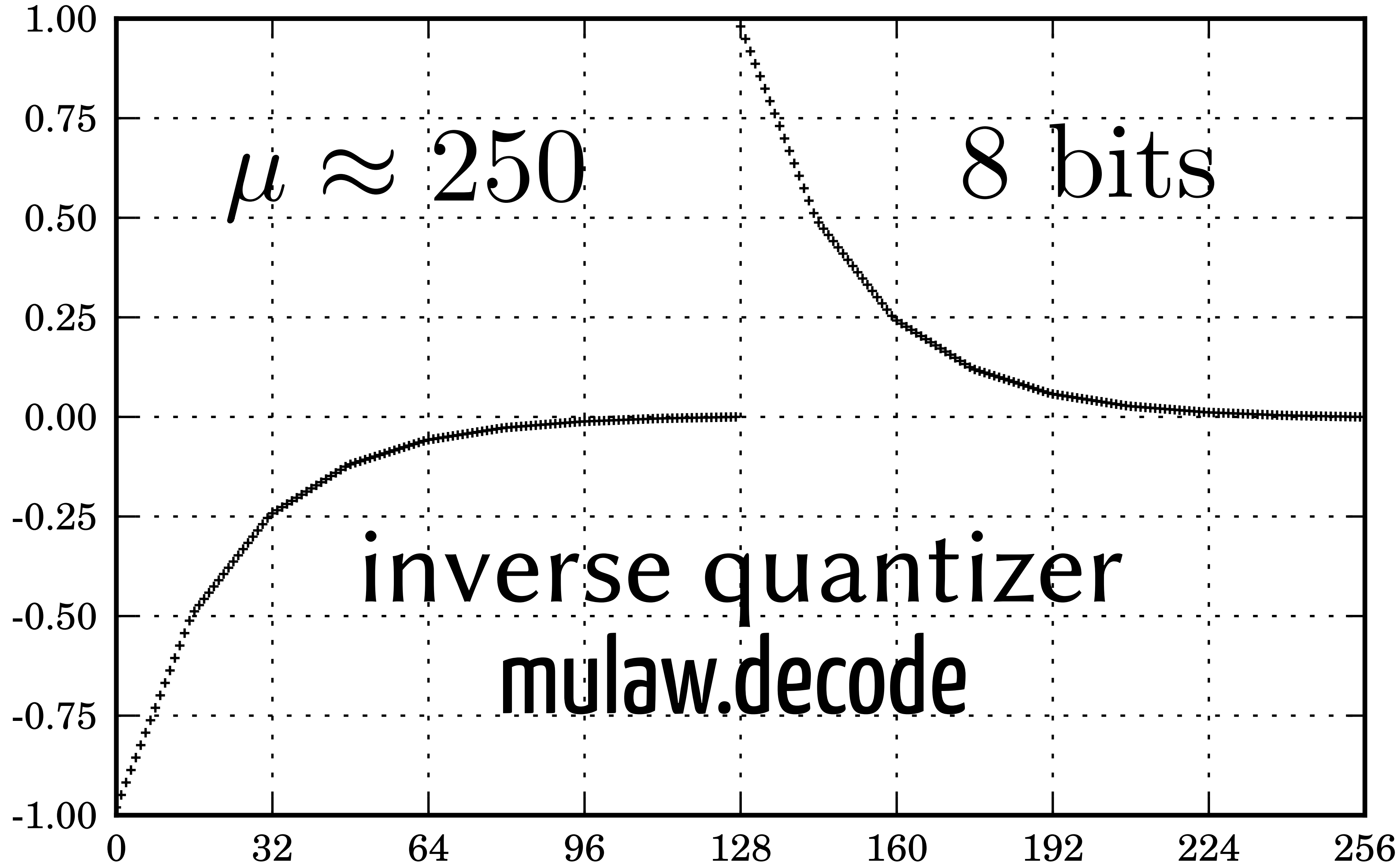
$$p(x) \propto \begin{cases} \dfrac{1}{1+\mu|x|} & \text{if } |x| \leq 1.0, \\ 0 & \text{otherwise.} \end{cases}$$

The optimal quantizer w.r.t. entropy such that $f([-1,1]) = [-1,1]$ is defined by:

$$f(x) = \operatorname{sgn}(x) \frac{\log(1+\mu|x|)}{\log(1+\mu)}$$

μ-law

Implements a piecewise approximation of f.
Part of the G.711 (ITU-T) standard.



Used in Sun/NeXT AU file format.