# Quantization

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Quantization is a process that maps a continous or discrete set of values into approximations that belong to a smaller set. Quantization is a lossy: some information about the original data is lost in the process. The key to a successful quantization is therefore the selection of an error criterion – such as entropy and signal-to-noise ratio – and the development of optimal quantizers for this criterion.

# Principles of Scalar Quantization

## Quantizers

A **scalar quantizer** $[\,\cdot\,]$ is an idempotent mapping from $\mathbb{R}$ to a countable subset of $\mathbb{R}$:

$$|\{[x]\,,\, x \in \mathbb{R}\}| \leq |\mathbb{N}| \;\text{ and }\; \forall\, x \in \mathbb{R},\; [[x]] = [x]$$

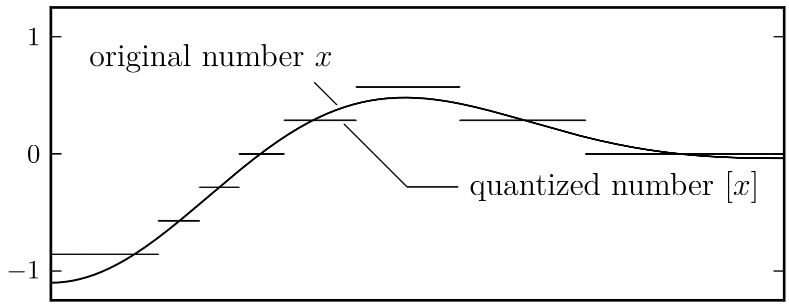



Figure 1: quantization of a time-varying value by a 4-bit midtread uniform quantization on $[-1.0, 1.0]$

This definition should be taken with a grain of salt as variants of the real line are often used, including the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$, the real line with signed zeros $\mathbb{R} \cup \{0^-, 0^+\}$, the real line plus the undefined symbol $\perp$, or a combination thereof.

The countability assumption is what makes the quantizer useful as an attempt to approximate a continous value by a discrete set that can be encoded as an integer. A quantizer is meant to be split into a **forward** and **inverse quantizer**: the forward quantizer builds from $x$ an integer code that refers to $[x]$ without ambiguity and the inverse quantizer builds the approximation $[x]$ back from the code.

Formally, a forward quantizer for $[\cdot]$ is a mapping $i[\,\cdot\,] : \mathbb{R} \to \mathbb{Z}$ such that $[x] = [y]$ implies $i[x] = i[y]$. Because of this property, $i[\,\cdot\,]$ may be factored into $i[\,\cdot\,] = i \circ [\,\cdot\,]$ where

$$i : \operatorname{rng} [\,\cdot\,] \to \mathbb{Z}.$$

The notation for the forward quantizer is therefore consistent with the use as $f[x]$ as a shortcut for $f([x])$. The associated inverse quantizer, denoted $i^{-1}$, is a left inverse of $i$: a mapping whose domain is a subset of $\mathbb{Z}$ that contains $\operatorname{rng} i$ and such that

$$\forall\, x \in \mathbb{R},\ (i^{-1} \circ i)[x] = [x]$$

The first step of this quantizer composition partitions the real line into the family of sets $(I_n)_n$ with

$$I_n = \{x \in \mathbb{R},\ i[x] = n\},\ n \in \operatorname{rng} i$$

The second step associates to any set into this partition a unique representative element. In every practical case we will encounter, the sets $I_n$ are – possibly unbounded – intervals, either open, half-open or closed. In this context, we associate to $x$ the decision values $[x]^-$ and $[x]^+$ to be

$$[x]^- = \inf \{y \in \mathbb{R},\ [x] = [y]\} \ \text{ and } \ [x]^+ = \sup \{y \in \mathbb{R},\ [x] = [y]\}$$

and the step of the quantization at point $x$ is

$$\Delta(x) = [x]^+ - [x]^-$$

**Example – Integer Rounding**

The floor function $\lfloor \cdot \rfloor$ is a scalar quantizer that maps a real number to the largest previous integer:

$$\forall\, x \in \mathbb{R},\ \lfloor x \rfloor \in \mathbb{Z} \ \text{ and } \ \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

A natural forward quantizer for $\lfloor \cdot \rfloor$ is … itself ! The identity $n \mapsto n$ is the corresponding inverse mapping. This quantizer partitions the real-line into the half-open intervals $I_n = [n, n+1)$ for any $i \in \mathbb{Z}$.

The `floor` function of NumPy is an finite-precision implementation of this function. Its argument and return value are (arrays of) 64-bits floating-point numbers.

To obtain a (forward) quantizer with a finite range indexable on 32 bits, we may modify the initial quantizer specification so that the data outside of the range $[-2^{31}, 2^{31}-1]$ – the range of 32-bit signed integers – is clipped:

$$\lfloor x \rfloor_{32} = \left| \begin{array}{rl} -2^{31} & \text{if } \ x \leq -2^{31} \\ 2^{31}-1 & \text{if } \ x \geq 2^{31}-1 \\ \lfloor x \rfloor & \text{otherwise.} \end{array} \right.$$

Given those modifications, a suitable finite implementation of the forward and inverse quantizers is the following `code/decode` pair:

```
from numpy import *

def encode(x):
    n = floor(x)
    n = clip(n, -2**31, 2**31 - 1)
    return int32(n)

def decode(n):
    return float64(n)

def quantize(x):
    return decode(encode(x))
```

The step function $\Delta$ of this quantization is defined by:

$$\Delta(x) = \left| \begin{array}{rl} +\infty & \text{if } \ x < -2^{31}+1 \\ 1 & \text{if } \ -2^{31}+1 \leq x < 2^{31}-1 \\ +\infty & \text{if } \ 2^{31}-1 \leq x \end{array} \right.$$

Other rounding functions may serve as the basis for similar schemes: the ceiling function $\lceil \cdot \rceil$ (NumPy function `ceil`) defined by:

$$\forall\, x \in \mathbb{R},\ \lceil x \rceil \in \mathbb{Z} \ \text{ and } \ \lceil x \rceil - 1 < x \leq \lceil x \rceil$$

Instead of selecting the lower or upper integer approximation of $x$ we may also select the nearest:

$$\forall\, x \in \mathbb{R},\ |x - [x]| = \min \{|x-n|,\, n \in \mathbb{Z}\}$$

The value $[x]$ is not defined by this relation when $x = n + 1/2$, $n$ being an integer. The NumPy function `round\_` rounds for example such real number to the nearest even integer.

This example suggests a general interface for quantizers. Such objects would provide an `encode` method for the forward quantization, a `decode` method for the inverse quantization and would be callable so that `quantizer(x)` would apply both steps to the data `x`. Such objects could inherit the following `Quantizer` base class:

```
class Quantizer(object):
    "Quantizers Base Class."
    def encode(self, data):
        raise NotImplementedError("undefined forward quantizer")

    def decode(self, data):
        raise NotImplementedError("undefined inverse quantizer")

    def __call__(self, data):
        return self.decode(self.encode(data))
```

We can then rewrite the above integer approximation quantizer as:

```
class RoundingQuantizer(Quantizer):
    def __init__(self, rounding=floor, integer_type=int32):
        self.rounding = rounding
        self.integer_type

    def encode(self, x):
        x = array(x)
        n = self.rounding(x)
        n = clip(n, -2**31, 2**31 - 1)
        return n.astype(self.integer_type)
```

```
    def decode(self, n):
        n = array(n)
        return n.astype(float64)

rounding_quantizer = RoundingQuantizer()
```

Note that this version of the quantizer is also vectorized: several values grouped in a NumPy array may be used as arguments to `encode` and `decode`. This is an implicit requirement that we expect all quantizer classes to follow for convenience.

## Uniform Quantization

A quantizer is uniform in an interval with lower bound $a$ and higher bound $b$ if its step function is constant in the interval. The size of the step is then directly connected to the width of the interval and the number $N$ of distinct values of $[x]$ by

$$\Delta(x) = \frac{b-a}{N}$$

The final option that characterizes the quantizer is the choice of the base rounding function. A reference implementation is then given by:

```
class Uniform(Quantizer):
   def __init__(self, low=0.0, high=1.0, N=2**8, rounding=round_):
        self.low = float(low)
        self.high = float(high)
        self.N = N
        self.delta = (high - low) / self.N
        self.rounding = rounding

    def encode(self, data):
        low, high, delta = self.low, self.high, self.delta
        data = clip(data, low + delta/2.0, high - delta/2)
        flints = self.rounding((data - low) / delta - 0.5)
        return array(flints, dtype=long)

    def decode(self, i):
        return self.low + (i + 0.5) * self.delta
```

Note that if the default value of `N` is selected – or more generally any even value – $[0] \neq 0$: the approximation error for 0 is not zero. When this property may be an issues, odd values of `N` may be selected – for example $2^8 - 1$ so that 0 is correctly approximated ; such a quantizer is called a **midtread** quantizer – opposed to the original **midrise** quantizer.

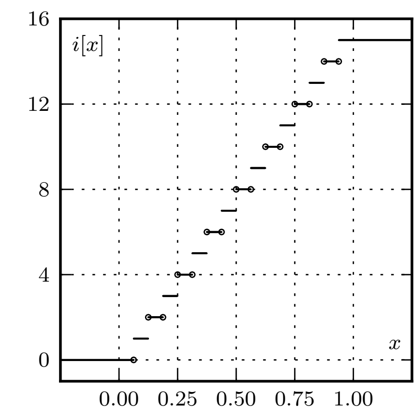



Figure 2: 4-bit uniform encoder on $(0, 1)$: forward quantizer

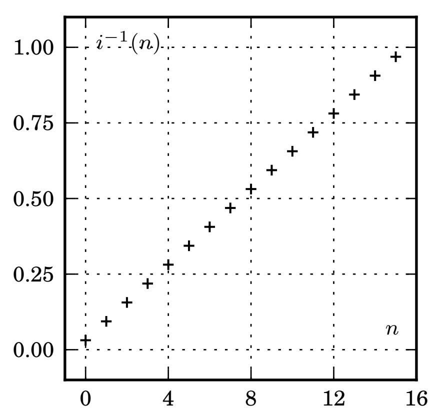



Figure 3: 4-bit uniform decoder on $(0, 1)$: inverse quantizer

### Quantization of Random Variables

Consider a random variable $X$ with values $x \in \mathbb{R}$ and a density of probability $p(x)$. For any $[x]$, we may consider the event $[X] = [x]$ with probability

$$P([X] = [x]) = \int_{\{y \in \mathbb{R},\ [y]=[x]\}} p(y)\, dy = \int_{[x]^-}^{[x]^+} p(y)\, dy$$

If the density $p$ is constant on every interval associated to the quantization, this equation may be simplified into:

$$P([X] = [x]) = p(x) \times \Delta(x)$$

More generally, if the quantizer values $[x]$ are dense enough – we say that the **high resolution assumption** is satisfied – then this relation holds approximately.

The entropy attached to this collection of events is maximal when every event is equally likely, that is, under this approximation, when the step $\Delta(x)$ is proportional to the inverse of $p(x)$

$$\Delta(x) \propto \frac{1}{p(x)}$$

### Implementation of Non-Uniform Quantizers

Non-uniform quantizers may be – at least conceptually – simply generated from uniform quantizers and non-linear transformations. If $[\,\cdot\,]$ denotes a uniform quantizer and $f$ is an increasing mapping, the function $[\,\cdot\,]_f$ defined by the equation

$$[x]_f = (f^{-1} \circ [\,\cdot\,] \circ f)(x)$$

and displayed in figure **??** is a nonlinear quantizer. The function $f$ is called the **characteristic function** of the quantizer. Depending on the selected range for the uniform quantizer, it is determined up to an affine transformation.

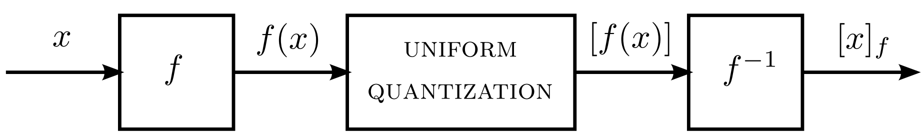



Figure 4: Nonlinear quantizer implementation

Note that if $f$ is linear or affine, that is $f(x) = ax + b$, the quantizer $[\,\cdot\,]_f$ is still uniform – that's a reason why uniform quantizers are sometimes called **linear quantizers**.

Let $\Delta$ be the step of the uniform quantizer et let's determine what quantization step $\Delta_f(x)$ is attached to this scheme.

For every value of $x$, the decision values attached to $y = f(x)$ by the uniform quantizer are $[y]^-$ and $[y]^+$. Hence, the decision values for $x$ and the non-linear quantization are

$$[x]_f^- = f^{-1}([y]^-) \text{ and } [x]_f^+ = f^{-1}([y]^+)$$

and if the high resolution assumption is satisfied the step $\Delta_f(x)$ is :

$$\Delta_f(x) = f^{-1}([y]^- + \Delta) - f^{-1}([y]^-) \simeq (f^{-1})'(f(x))\Delta = \frac{\Delta}{f'(x)}$$

something that is remembered as

$$\Delta_f(x) \propto \frac{1}{f'(x)}$$

The proportionaly constant may be easily recovered by noting that when $f(x) = x$, $[\,\cdot\,]_f = [\,\cdot\,]$ and therefore $\Delta(x) = \Delta$. If we impose moreover $f(0) = 0$, we find

$$f(x) \propto \int_0^x \frac{ds}{\Delta(s)}$$

If the quantizer is to maximize the entropy for the random variable $X$ with density $p(x)$ we obtain

$$f(x) \propto \int_0^x p(y)\, dy$$

**Example**

Let's consider the digital audio signal displayed in figure **??**.

The uniform quantization on $(-1, 1)$ with step $\Delta = 10^{-1}$ is dense enough so that the associated histogram may be considered as a continuous function of the parameter $x$. We observe in figure **??** that this partition generates – for a large range of values of $x$ – a counting measure $n(x)$ of a few thousands. The ratio $n(x)/n$ where $n$ is the total number of samples should therefore generate a good approximation of the density of the signal, considered as a sequence of independent and identically distributed values.

The logarithm of the histogram is similar to a function of the type $-a|x|+b$, $a > 0$ (cf fig. **??**). We therefore select $p(x) \propto \exp(-a|x|)$. The optimal quantization – for the entropy criterion – and the corresponding characteristic function $f$ such that $f(0) = 0$ are therefore given by:

$$\Delta(x) \propto e^{a|x|} \text{ and } f(x) \propto \operatorname{sign}(x)(1 - e^{-a|x|})$$

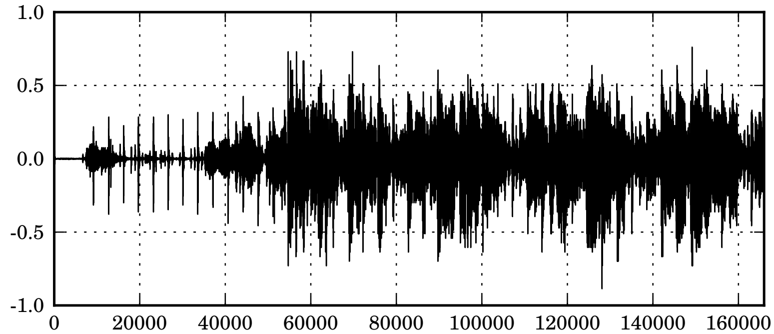



Figure 5: Around 20 seconds of audio data

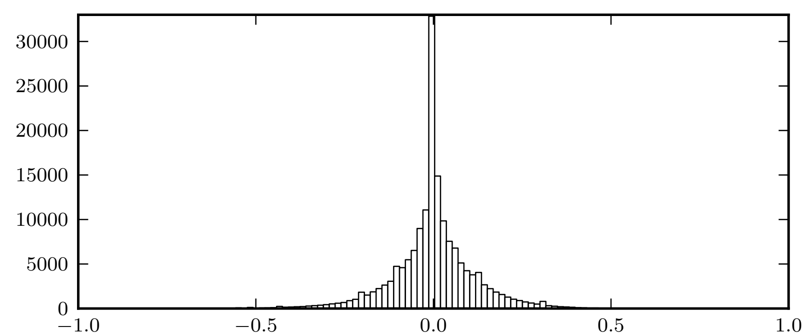



Figure 6: Audio Data Histogram

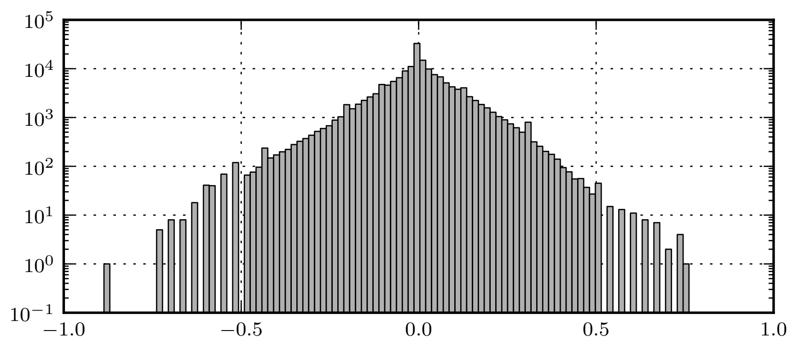



Figure 7: Log plot of the audio data histogram

# Logarithmic Quantization

We consider in this section several related quantizers whose characteristic function is – roughly speaking – the logarithm of their argument.

## The $\mu$-law Quantizer

Consider the probability law

$$p(x) \propto \left| \begin{array}{ll} \dfrac{1}{1+\mu|x|/A} & \text{if } |x| \leq A, \\ 0 & \text{otherwise.} \end{array} \right.$$

The threshold $A$ is necessary as otherwise the right-hand side of the equation would not be summable. The parameter $a$ controls directly the relative probability of low and high amplitude values as $p(\pm A)/p(0) = 1/(1+\mu)$. In the limit case $\mu = 0$, we end up with a uniform probability distribution on $[-A, A]$.

The optimal quantizer for the entropy criterion satisfies () and therefore the characteristic function $f$ such that $f(0) = 0$ satisfies

$$f(x) \propto \operatorname{sign}(x) \ln \left(1 + \mu \frac{x}{A}\right).$$

If we limit the range of the quantizer to $[-1, 1]$ (we set $A = 1$) and enforce the constraint $f([-1, 1]) = [-1, 1]$, we end up with

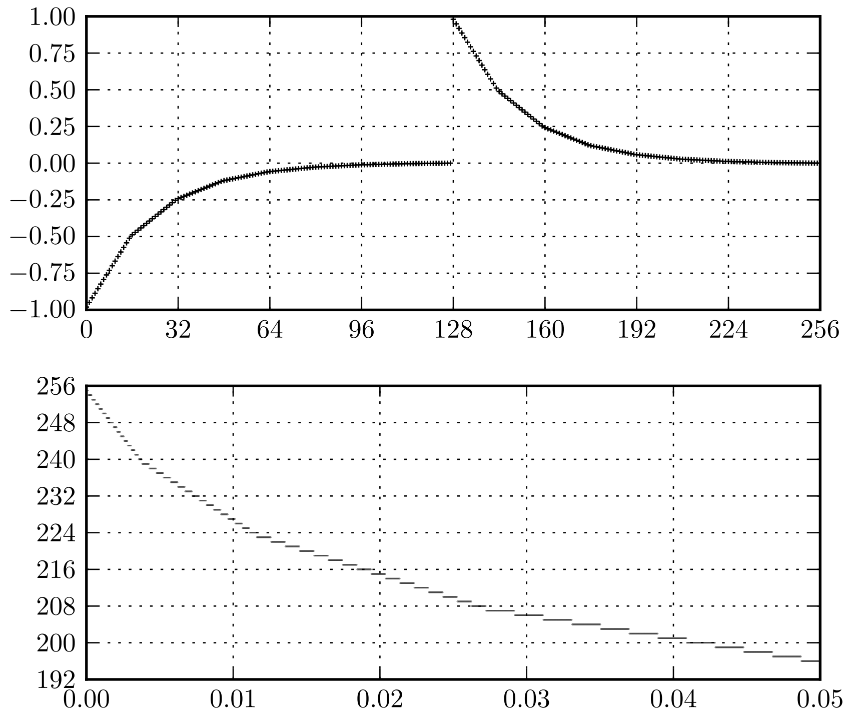
$$f(x) = \operatorname{sign}(x) \frac{\log(1 + \mu|x|)}{\log(1+\mu)}$$

This quantization scheme is called $\mu$-law and is for example used in the NeXT/Sun AU audio file format (files with extension `.au` or `.snd`). The actual implementation of the law, specified in the ITU-T G.711 standard – differs slightly from the theoretical formulas. A reference implementation is given in the code below:

```
class MuLaw(Quantizer):
    """
    Mu-law quantizer
    """
    scale  = 32768
    iscale = 1.0 / scale
    bias   = 132
    clip   = 32635
    etab   = array([0, 132, 396, 924, 1980, 4092, 8316, 16764])

    @staticmethod
```

```
    def sign(data):
        """
        Sign function such that sign(+0) = 1 and sign(-0) = -1
        """
        data = array(data, dtype=float)
        s = numpy_sign(data)
        i = where(s==0)[0]
        s[i] = numpy_sign(1.0 / data[i])
        return s


    def encode(self, data):
        data = array(data)
        s = MuLaw.scale * data
        s = minimum(abs(s), MuLaw.clip)
        [f,e] = frexp(s + MuLaw.bias)

        step  = floor(32*f) - 16    # 4 bits
        chord = e - 8               # 3 bits
        sgn   = (MuLaw.sign(data) == 1)   # 1 bit

        mu = 16 * chord + step      # 7-bit coding
        mu = 127 - mu               # bits inversion
        mu = 128 * sgn + mu         # final 8-bit coding

        return array(mu, dtype=uint8)

    def decode(self, i):
        i = array(i)
        i = 255 - i
        sgn = i > 127
        e = array(floor(i / 16.0) - 8 * sgn + 1, dtype=uint8)
        f = i % 16
        data = ldexp(f, e + 2)
        e = MuLaw.etab[e-1]
        data = MuLaw.iscale * (1 - 2 * sgn) * (e + data)

        return data

mulaw = MuLaw()
```

Note that this code is applied to values between $-1$ and 1 and uses 8 bits. The most significant bit encodes the sign; the amplitude of the signal is coded by the 7 remaining bits. The effective value of $\mu$ is approximately 250 but instead of using the expression $\log(1+\mu|x|)$, we prefer a piecewise affine approximation of it (see fig **??**). The values $[x]$ are then all multiples of $2^{-13}$ which limits the additional quantization error when the original signal is initially encoded with a uniform law using 14 bits or more. To ease the error correction when transmitted the bits other than the sign bit are finally inverted.




## IEEE754 Floating-Point Numbers and $A$-law

All scientific computing applications use implicitely a quantizer: the quantizer that represents approximation of real numbers in the floating-point arithmetic. The description of two types of numbers – `single` and `double` (or rather, single and double-precision numbers) – is detailled in the IEEE 754 standard. In both cases, 1 bit is allocated to code the sign of the number, $m$ bits for the exponent part and $n$ bits for the fraction part,

$$s \in \{0, 1\},\ e \in \{0, \cdots, 2^m - 1\},\ f \in \{0, \cdots, 2^n - 1\}$$

consequently any real number is represented by an integer in $\{0, \cdots, 2^{m+n+1}\}$ according to:

$$n = s \times 2^{m+n} + e \times 2^n + f \in \{0, \cdots, 2^{m+n+1}\}$$

The `single` type is defined by $(m, n) = (8, 23)$ and the `double` type by $(m, n) = (11, 52)$ ; they are respectively coded on 32 and 64 bits.

We define

$$e_0 = 2^{m-1} - 1$$

so that the value of the actual exponent $e - e_0$ range (almost symmetrically) from $2^{m-1}$ to $-2^{m-1} + 1$. The inverse quantizer attached to the standard floating point number representation is defined as follows: for an integer $n$, $[x] = i^{-1}(n)$ is given by

$$[x] = \left| \begin{array}{rc} NaN & \text{if } e = 2^m - 1 \text{ and } f \neq 0 \\ (-1)^s \infty & \text{if } e = 2^m - 1 \text{ and } f = 0 \\ (-1)^s (1 + f/2^n) \times 2^{e-e_0} & \text{if } \ 0 < e < 2^m - 1 \\ (-1)^s (f/2^n) \times 2^{1-e_0} & \text{if } \ e = 0 \end{array} \right.$$

The structure of theses inverse quantizers are displayed in the figure **??**; they are piecewise affine approximation of an exponential with a base of 2, except in the range $e = 0$ (the so-called **denormalized numbers**) where the graph is linear.

[graph of the inverse quantizer for a floating point representation such that $(m, n) = (4, 3)$] (images/float.pdf)

The $A$-law is a variant of the $\mu$-law that has a structure similar the `single` and `double` types of floating point arithmetic but with a base different from 2. Given a value of $A$ (often 87.7), the inverse of its characteristic function is defined on $[-1, 1]$ by

$$f^{-1}(x) = \operatorname{sgn}(x) \times \left| \begin{array}{rl} (1 + \ln A)|x|/A & \text{if } \ |x| < \frac{1}{1 + \ln A} \\ \exp(x(1 + \ln A) - 1)/A & \text{otherwise.} \end{array} \right.$$

# Signal-to-Noise Ratio

## Computation of the signal-to-noise ratio

For a given sequence of $k$ values $x_n$, the output $[x_n]$ of a quantizer may be interpreted as the sum of the original value and a perturbation sequence $b_n = [x_n] - x_n$ called a **noise**. The square of the signal-to-noise ratio – or SNR – is simply the ratio between the energies of those two values:

$$\text{SNR}^2 = \frac{\mathbb{E}sp \left( \displaystyle\sum_{n=0}^{k-1} x_n^2 \right)}{\mathbb{E}sp \left( \displaystyle\sum_{n=0}^{k-1} b_n^2 \right)}$$

The SNR is often measured in **decibels** (dB):

$$\text{SNR [dB]} = 20 \log_{10} \text{SNR} = 10 \log_{10} \text{SNR}^2$$

When the values $x_n$ are independent and follow the same probability law $p(x)$, this energy is given by

$$\mathbb{E}\left(\sum_{n=0}^{p-1} x_n^2\right) = k\,\mathbb{E}\left(x_n^2\right) = k\int_{-\infty}^{+\infty} x^2 p(x)\,dx$$

and under a high resolution assumption we have

$$\begin{aligned}
\mathbb{E}sp(b_n^2) &= \int_{-\infty}^{+\infty} ([x]-x)^2 p(x)\,dx \\
&= \sum_y \int_y^{y+\Delta(y)} ([x]-x)^2 p(x)\,dx \\
&\simeq \sum_y p(y) \int_y^{y+\Delta(y)} (y+\Delta(y)/2-x)^2\,dx \\
&= \sum_y p(y)\frac{\Delta(y)^3}{12} \\
&\simeq \sum_y \int_y^{y+\Delta(y)} \frac{\Delta(x)^2}{12} p(x)\,dx \\
&= \int_{-\infty}^{+\infty} \frac{\Delta(x)^2}{12} p(x)\,dx \\
&= \frac{1}{12}\mathbb{E}sp\left(\Delta(x_n)^2\right)
\end{aligned}$$

Finally

$$\mathrm{SNR}^2 = 12\frac{\mathbb{E}sp(x_n^2)}{\mathbb{E}sp(\Delta(x_n)^2)} = 12\frac{\displaystyle\int_{\mathbb{R}} x^2 p(x)\,dx}{\displaystyle\int_{\mathbb{R}} \Delta(x)^2 p(x)\,dx}$$

In the typical case where the probability density of the signal is uniform on $[-A, A]$ and the quantization is uniform on this range with a step $\Delta$, we end up with

$$\mathrm{SNR} = 2A/\Delta$$

**Maximization of the SNR**

For a given density of probability, how can we select the quantization scheme so that the SNR is maximal ? Formulated like that, this problem is not well-posed because the quantization noise may be made a small as possible with a decrease of the quantization step. The significant problem is to solve this problem under a constant bit budget. Without any loss of generality, we may assume that the signal has values in $[-1, 1]$ and that the characteristic function of the searched quantization satisfies $f([-1, 1]) = [-1, 1]$. If we allocate $N$ bits to the quantization scheme, the step $\Delta(x)$ is determined by

$$\Delta(x) = \frac{2^{-N+1}}{f'(x)}$$

The SNR then takes the form

$$\mathrm{SNR} = \kappa 2^N$$

where the value of $\kappa$ depend only from the probability law of the signal and of the choice of $f$. In decibels, this equation is written as

$$\mathrm{SNR}\ [\mathrm{dB}] \simeq 6.02 \times N + \kappa'$$

that is, every extra bit increase the SNR by approximately 6 dB. To maximize the SNR, we then have to solve

$$\min_{f'} \int_{-1}^{1} \frac{1}{f'(x)^2} p(x)\, dx \ \text{ subject to } \ f(1) - f(-1) = 2$$

or even, with $\psi = f'$

$$\min_{\psi} J(\psi) = \int_{-1}^{1} \frac{1}{\psi(x)^2} p(x)\, dx \ \text{ with } \ K(\psi) = \int_{-1}^{1} \psi(x)\, dx = 2$$

At the optimum, there is a $\lambda \in \mathbb{R}$ such that the lagrangian $L(\psi) = J(\psi) + \lambda K(\psi)$ satisfies $dL(\psi) = 0$, that is

$$\text{for all } \ \delta\psi : [-1, 1] \to \mathbb{R}, \ \int_{-1}^{1} \left( -\frac{2}{\psi(x)^3} p(x) + \lambda \right) (\delta\psi)(x)\, dx = 0$$

and that implies

$$-\frac{2}{\psi(x)^3} p(x) + \lambda = 0$$

and hence

$$f'(x) \propto (p(x))^{\frac{1}{3}}.$$