Boundary Variations in the Navier-Stokes Equations and Lagrangian Functionals

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Abstract

We study the shape sensitivity of the stationary Navier-Stokes Equations in the general case of non-homogeneous and shape-dependent forces and boundary conditions.

Under an assumption of non-singularity of the equations, the shape differentiability of the velocity and the pressure are obtained in some Sobolev spaces. The influence of the regularity of the geometrical and functional data on the best space for which the result holds is stressed.

We apply these results on a class of shape functionals where a high regularity is required : the lagrangian functionals. Their main characteristic is to take into account the paths of the fluid particles. The usual shape calculus is extended to take into account such features. We determine the shape derivative of a shape-dependent flow and develop the methods to achieve the explicit calculation of the shape gradient.

1 Introduction

1.1 The Navier-Stokes Equations

We consider the stationary incompressible Navier-Stokes Equations (NSE) in some smooth enough open and bounded sets $\Omega \subset \mathbb{R}^3$ of boundary Γ

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad & \text{in } \Omega \\ \operatorname{div} u = 0 \quad & \text{in } \Omega \\ u = g \quad & \text{on } \Gamma \end{aligned} \tag{1}$$

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The shape analysis will therefore include as special cases the situation where the flow is uniquely driven by the force field (with homogeneous Dirichlet boundary conditions, cf. [3]), as well as the one where the flow is induced by a body moving at a constant velocity (cf. [1]).

Moreover, for a greater generality, we assume that the data (f, g) may explicitly depend on the shape Ω. For a given set Ω, the corresponding value of f, denoted f_Ω, is *a priori* defined only on Ω (and g_Γ only on Γ). We assume that $f_\Omega \in H^{-1}(\Omega;\mathbb{R}^3)$, $g_\Gamma \in H^{1/2}(\Gamma;\mathbb{R}^3)$ and moreover that for any admissible set Ω, and for any connected component Λ of Γ, we have

$$\int_\Lambda \langle g_\Gamma, n\rangle_{\mathbb{R}^3}\, d\mathcal{H}^2 = 0 \tag{2}$$

(n_Ω, or simply n when no doubt is possible, is the unit outer normal to Ω).

We recall briefly the corresponding abstract setting. We denote by $V^1(\Omega)$ the space $\{u \in H^1(\Omega;\mathbb{R}^3),\ \operatorname{div} u = 0\}$, we set $V_0^1(\Omega) = V^1(\Omega) \cap H_0^1(\Omega;\mathbb{R}^3)$ and $V^{1/2}(\Gamma) = \{g \in H^{1/2}(\Gamma;\mathbb{R}^3),\ \int_\Gamma \langle g, n\rangle\, d\mathcal{H}^2 = 0\}$. The linear operator $\pi : H^{-1}(\Omega;\mathbb{R}^3) \to (V_0^1(\Omega))'$ is Leray's projector: $\pi(f)$ is the restriction of the linear form f on $H_0^1(\Omega;\mathbb{R}^3)$ to $V_0^1(\Omega)$. For any u and $v \in V^1(\Omega)$, we set $Au = -\pi(\Delta u)$ and $B(u,v) = \pi((u\cdot\nabla)v)$. The (nonlinear) Navier-Stokes operator is the mapping

$$F: \begin{array}{ccc} V^1(\Omega) & \to & V_0^1(\Omega)' \times V^{1/2}(\Gamma) \\ u & \mapsto & (\nu Au + B(u,u), u|_\Gamma) \end{array} \tag{3}$$

1.2 Shape Analysis framework

The shape sensitivity of this equation is studied in the framework of the *Speed Method*. The first step is to generate some of the geometries around the reference set Ω while staying in a given design region. In that purpose, we associate to a smooth, open and bounded set D (designed later on as the *hold-all*), a velocity space $\mathcal{V}$, chosen among the $\mathcal{V}_k$ for a $k \geq 1$

$$\mathcal{V}_k = \{V \in \mathcal{C}^0([-T;T];\mathcal{C}^k(\overline{D};\mathbb{R}^3)),\ \langle V, n_D\rangle_{\mathbb{R}^3} = 0 \text{ on } \partial D\} \tag{4}$$

Then for any $V \in \mathcal{V}$, a one-parameter family of deformations $T_s : \overline{D} \to \overline{D}$ is given by the following initial-value problem

$$\begin{array}{l} \partial_s T_s = V(s) \circ T_s \\ T_0 = I \end{array} \tag{5}$$

We denote $\Omega_s := T_s(\Omega)$ the corresponding transported sets generated by Ω and V. We also set $\Gamma_s := T_s(\Gamma)$.

The regularity of shape-dependent mappings such as f and g are defined in the following way. Let $W(\Omega)$ be either one of the Sobolev spaces $W^{m,p}(\Omega;\mathbb{R}^m)$ ($m \geq 0$, $p \geq 1$) or one of the spaces $\mathcal{C}^k(\overline{\Omega};\mathbb{R}^m)$ ($k \geq 0$).

Definition 1 *We say that the mapping f is $\mathcal{C}^k$ with respect to the shape in $W(\Omega)$ if for any $V \in \mathcal{V}$, the mapping $s \mapsto f_{\Omega_s} \circ T_s$ is in $\mathcal{C}^k(I;W(\Omega))$ on a neighbourhood I of* 0.

The shape sensitivity results of the section 2 are expressed in Sobolev spaces whereas the $\mathcal{C}^k$-spaces are needed for the study of the lagrangian functionals (section 3).

Of course, an analogous definition holds for the shape-dependent mappings g defined on the boundary Γ. Two different types of derivatives with respect to the geometry may be used to describe the variations of these fields : the *material derivative* of f and g, given by

$$\dot{f}_\Omega = \frac{d}{ds} f_{\Omega_s} \circ T_s|_{s=0} \quad \text{and} \quad \dot{g}_\Gamma = \frac{d}{ds} g_{\Gamma_s} \circ T_s|_{s=0} \tag{6}$$

and the *shape derivative* and *boundary shape derivative*

$$f'_\Omega = \dot{f}_\Omega - Df_\Omega \cdot V(0) \quad \text{and} \quad g'_\Gamma = \dot{g}_\Gamma - D_\tau g_\Gamma \cdot V(0) \tag{7}$$

($D_\tau g$ is the tangential Jacobian matrix of g). When the choice of Ω is clear, the corresponding subscript may be dropped.

In order to characterize the regularity of shape-dependent forces f that belong to $H^{-1}(\Omega;\mathbb{R}^3)$, we define $\phi_s : H^1_0(\Omega;\mathbb{R}^3) \to H^1_0(\Omega_s;\mathbb{R}^3)$ by $\phi_s(f) = f \circ T_s^{-1}$ and set $\gamma_s = \det DT_s$ (see also section 2.2). Then for $W(\Omega) = H^{-1}(\Omega;\mathbb{R}^3)$, $s \mapsto f_{\Omega_s} \circ T_s$ has to be replaced by $s \mapsto \gamma_s^{-1}\phi_s^\star(f_{\Omega_s})$ in the definition 1 as well as in the definition of the material derivative (formula (6)). This extension of the definition is consistent : these two expressions are equal when f_Ω belongs to $L^2(\Omega;\mathbb{R}^3)$. The shape derivative (when it exist) is still given by the equation (7).

2 Sensitivity Results

In this section, we derive a result on the regularity of the solution (u,p) of the Navier-Stokes Equations with respect to a perturbation of the boundary. The first part is dedicated to the description of the set of assumptions and of the statement of the result. The corresponding proof is developed in the sections (2.2) to (2.4).

2.1 Assumptions and Main Statement

First, we describe the regularity needed on the geometrical and functional data to derive our sensitivity result. The degree of regularity is given, in this assumption as well as in the proposition by an integer $k \geq 1$.

($\mathcal{A}_1$: Regularity of the data)

• Ω and D are $\mathcal{C}^{k+1}$ and the velocity space $\mathcal{V}$ is $\mathcal{V}_{k+2}$.

• the mappings f and g are continuous with respect to the shape in $H^{k-1}(\Omega; \mathbb{R}^3)$ and $H^{k+1/2}(\Gamma; \mathbb{R}^3)$ respectively.

• the mappings f and g are continuously differentiable with respect to the shape in $H^{k-2}(\Omega; \mathbb{R}^3)$ and $H^{k-1/2}(\Gamma; \mathbb{R}^3)$ respectively.

Remark 1 This set of assumptions is rather designed to handle the high regularity case, even if the methods developed in the following sections could be used as well to study the situations where such a regularity is not available.

The smoothness of the geometry is determined so that the regularity of the solutions of the Navier-Stokes Equations is maximal with respect to the known regularity of f and g.

The assumptions made for these mappings may be surprising at first: the spatial regularity of f and $\dot{f}$ on one hand, of g and $\dot{g}$ on the other hand, are not the same. However this is the usual situation; when f does not depend on Ω, that is when there is for any $V \in \mathcal{V}$ a $F \in H^k(D; \mathbb{R}^3)$ such that $f_{\Omega_s} = F|_{\Omega_s}$, the existence of $\dot{f}$ takes place *a priori* only in H^{k-1}, the material derivative being given by $\dot{f} = \langle D_x f, V(0) \rangle$. An analogous property holds for g.

An interesting consequence of that gap is that the material and shape derivatives of the data have the same spatial regularity as

$$f'_\Omega = \dot{f}_\Omega - Df_\Omega \cdot V(0) \text{ and } g'_\Gamma = \dot{g}_\Gamma - D_\tau g_\Gamma \cdot V(0)$$

The theorem 1 shows that the solutions u and p of the Navier-Stokes Equations exhibit the same kind of regularity. Therefore, the same property holds for their derivatives with respect to the shape.

The second assumption, less common, results from the non-linearity of the Navier-Stokes Equations: it has no equivalent for the linear shape optimization problems as the linearized problem is automatically well-posed when the initial one is.

($\mathcal{A}_2$: Non-Singularity of the NSE) The couple (f_Ω, g_Γ) is a regular value of the Navier-Stokes operator (3): for any solution u of the system (1), the operator $DF(u)$ is an isomorphism.

Remark 2 Explicitly, the assumption $\mathcal{A}_2$ states that for any $k \in H^{-1}(\Omega;\mathbb{R}^3)$, $l \in H^{1/2}(\Gamma;\mathbb{R}^3)$ such that $\int_\Gamma \langle k, n\rangle \, d\mathcal{H}^2 = 0$ and any solution of the Navier-Stokes Equations, the system

$$\begin{aligned} -\nu\Delta v + (u\cdot\nabla)v + (v\cdot\nabla)u + \nabla q = h \quad & \text{in } \Omega \\ \operatorname{div} v = 0 \quad & \text{in } \Omega \\ v = k \quad & \text{on } \Gamma \end{aligned} \tag{8}$$

has a unique solution $(v,q) \in H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})/\mathbb{R}$.

This assumption is naturally fulfilled in the high viscosity (or small data) case: when ν large enough with respect to the data (f,g) or conversely when f and g are small enough in $H^1(\Omega;\mathbb{R}^3)$ and $H^{1/2}(\Gamma;\mathbb{R}^3)$ for a given viscosity ν, then the linearized Navier-Stokes Equations at the considered solution are well-posed.

In general, without such an assumption on the viscosity, the Navier-Stokes operator is still *generically* non-singular with respect to (f,g) or even with respect to g for a fixed value of f (see [6], [9]). However, the well-posedness of the linearized problem is ensured only in more regular (Hölder or Sobolev) spaces than those considered in the section 1.1.

We may now state the main result of this section:

Theorem 1 *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are satisfied. Let (u,p) be a solution of the Navier-Stokes Equations in Ω. Then, for any $V \in \mathcal{V}$, there is a neighbourhood I of 0 in $\mathbb{R}$ and a locally unique family $s \in I \mapsto (u_s, p_s)$ of solutions of the Navier-Stokes Equations in Ω_s such that:*

(i) Initial Condition: $(u_0, p_0) = (u, p)$

(ii) Regularity: $(\mathcal{U}, \mathcal{P}) : [s \mapsto (u_s \circ T_s, p_s \circ T_s)]$ *is such that*

- $\mathcal{U} \in \mathcal{C}^0(I; H^{k+1}(\Omega;\mathbb{R}^3)) \cap \mathcal{C}^1(I; H^k(\Omega;\mathbb{R}^3))$
- $\mathcal{P} \in \mathcal{C}^0(I; H^k(\Omega;\mathbb{R}^3)/\mathbb{R}) \cap \mathcal{C}^1(I; H^{k-1}(\Omega;\mathbb{R}^3)/\mathbb{R})$

(iii) Shape Derivatives: u' *and* p' *are the solutions of*

$$\begin{aligned} -\nu\Delta u' + (u\cdot\nabla)u' + (u'\cdot\nabla)u + \nabla p' = f' \quad & \text{in } \Omega \\ \operatorname{div} u' = 0 \quad & \text{in } \Omega \\ u' = -\frac{\partial u}{\partial n}\langle V(0), n\rangle + g'_\Gamma \quad & \text{on } \Gamma \end{aligned} \tag{9}$$

2.2 Transport

We develop a modification of the process of *transported equation* which is used to characterize the regularity with respect to the shape of the solutions of a

PDE problem. Usually, we associate to any $V \in \mathcal{V}$ the family of one-to-one correspondences $(\phi_s)_{s\in\mathbb{R}}$

$$\phi_s \begin{pmatrix} W(\Omega) & \to & W(\Omega_s) \\ u & \mapsto & u \circ T_s^{-1} \end{pmatrix} \tag{10}$$

where $W(\Omega)$ and $W(\Omega_s)$ are the adequate functional spaces for the considered problem) and we study the properties of the problem satisfied by $\phi_s^{-1}(u_{\Omega_s})$ where u_{Ω_s} is the solution of the initial problem in Ω_s; this transported problem is expressed in the fixed space Ω. However, such an approach is not straightforward here *because the transport of the compatibility condition (2) would not* in general *lead to fixed functional spaces*: this property is directly true for homogeneous Dirichlet boundary conditions or with a slight adaptation for the non-homogeneous case $u|_\Gamma = g$ *when* g *does not depend on the shape* Γ (cf. [1]). However, this does not include the general case.

In order to circumvent that problem, we introduce the Piola transformations $\psi_s : L^2(\Omega;\mathbb{R}^3) \to L^2(\Omega_s;\mathbb{R}^3)$, defined by :

$$\psi_s(u) = \phi_s(\gamma_s^{-1} J_s \cdot u) \tag{11}$$

with $J_s = DT_s$ and $\gamma_s = \det(J_s)$. These coefficients satisfy the following lemma.

Lemma 1 *For any $V \in \mathcal{V}_k$, and any $s \in \mathbb{R}$, J_s and γ_s are invertible. Moreover, J_s and J_s^{-1} are in $\mathcal{C}^1(\mathbb{R};\mathcal{C}^{k-1}(\overline{D};\mathbb{R}^{3\times 3}))$ and γ_s and γ_s^{-1} are in $\mathcal{C}^1(\mathbb{R};\mathcal{C}^{k-1}(\overline{D};\mathbb{R}))$.*

The proof is a direct consequence of the regularity of $s \mapsto T_s$ described in [10]. Most of the classical properties of the ϕ_s are also satisfied by the ψ_s. For any $k \geq 1$, $V \in \mathcal{V}_{k+1}$ and $s \in \mathbb{R}$,

• ϕ_s and ψ_s are isomorphisms between $L^2(\Omega;\mathbb{R}^3)$ and $L^2(\Omega_s;\mathbb{R}^3)$; their inverse are given by:

$$\phi_s^{-1}(u) = u \circ T_s \ \text{ and } \ \psi_s^{-1}(u) = \gamma_s J_s^{-1} \phi_s^{-1}(u)$$

• The regularity of the mappings are preserved through the transport: for any integer $m \leq k$, ϕ_s and ψ_s induce isomorphisms between $H^m(\Omega;\mathbb{R}^3)$ and $H^m(\Omega_s;\mathbb{R}^3)$. As $\mathcal{D}^m(\Omega)$ is also mapped into $\mathcal{D}^m(\Omega_s)$, the adjoint operators $\phi_s^\star$ and $\psi_s^\star$ are isomorphisms from $H^{-m}(\Omega_s;\mathbb{R}^3)$ to $H^{-m}(\Omega;\mathbb{R}^3)$.

• the same property hold for boundary Sobolev spaces: for any integer m such that $m + 1/2 \leq k$, ϕ_s and ψ_s induce isomorphisms between $H^{m+1/2}(\Gamma;\mathbb{R}^3)$ and $H^{m+1/2}(\Gamma_s;\mathbb{R}^3)$.

Moreover, ψ_s satisfies the two extra following properties:

Proposition 1 *Let $V \in \mathcal{V}_2$ and $s \in \mathbb{R}$. For any $u \in L^2(\Omega;\mathbb{R}^3)$ and $\varphi \in H_0^1(\Omega;\mathbb{R}^3)$, we have*

$$\langle \operatorname{div} u, \varphi \rangle_{H^{-1}\times H_0^1} = \langle \operatorname{div}(\psi_s u), \phi_s \varphi \rangle_{H^{-1}\times H_0^1} \tag{12}$$

Therefore ψ_s also induce an isomorphism between $V^1(\Omega)$ and $V^1(\Omega_s)$. Moreover, for any $u \in H^1(\Omega;\mathbb{R}^3)$ and $\varphi \in H^1(\Omega;\mathbb{R}^3)$,

$$\int_{\Gamma_s} \langle \psi_s(u), n_{\Omega_s} \rangle_{\mathbb{R}^3} \, \phi_s(\varphi) \, d\mathcal{H}^2 = \int_{\Gamma} \langle u, n_\Omega \rangle_{\mathbb{R}^3} \, \varphi \, d\mathcal{H}^2 \tag{13}$$

Proof – By a change of variable, we have the equality

$$\begin{aligned}
\langle \operatorname{div} \psi_s(u), \phi_s(\varphi) \rangle_{H^{-1}\times H_0^1} &= -\int_{\Omega_s} \langle \psi_s(u), \nabla(\phi_s(\varphi)) \rangle \, dx \\
&= -\int_{\Omega} \Big\langle \psi_s(u) \circ T_s, \nabla(\phi_s(\varphi)) \circ T_s \Big\rangle \, \gamma_s \, dx
\end{aligned}$$

As $\nabla(\phi_s(\varphi)) \circ T_s = \nabla(\varphi \circ T_s^{-1}) \circ T_s = [DT_s^*]^{-1} \nabla \varphi$, we obtain as desired

$$\begin{aligned}
\langle \operatorname{div} \psi_s(u), \phi_s(\varphi) \rangle_{H^{-1}\times H_0^1} &= -\int_{\Omega} \Big\langle \gamma_s J_s^{-1} \psi_s(u) \circ T_s, \nabla \varphi \Big\rangle \, dx \\
&= -\int_{\Omega} \langle u, \nabla \varphi \rangle \, dx \\
&= \langle \operatorname{div} u, \varphi \rangle_{H^{-1}\times H_0^1}
\end{aligned}$$

To establish the equality 13, we make the same change of variable. It yields

$$\int_{\Gamma_s} \langle \psi_s(u), n_{\Omega_s} \rangle \, \phi_s(\varphi) \, d\mathcal{H}^2 \;=\; \int_{\Gamma} \langle \psi_s(u) \circ T_s, n_{\Omega_s} \circ T_s \rangle \, (\phi_s(\varphi) \circ T_s) \, \omega_s \, d\mathcal{H}^2$$

with $\omega_s = \gamma_s \|(J_s^*)^{-1} n_\Omega\|$. As $n_{\Omega_s} \circ T_s = \dfrac{(J_s^*)^{-1} n_\Omega}{\|(J_s^*)^{-1} n_\Omega\|}$, we finally obtain

$$\int_{\Gamma_s} \langle \psi_s(u), n_{\Omega_s} \rangle \, \phi_s(\varphi) \, d\mathcal{H}^2 \;=\; \int_{\Gamma} \langle u, n_\Omega \rangle \, \varphi \, d\mathcal{H}^2$$

■

Remark 3 In the process of definition if the material derivative, described in the section 1.1, we could replace the mappings ϕ_s by the ψ_s and therefore defined the *Piola material derivative* of f, $\dot{f}^P$, by

$$\dot{f}_\Omega^P = \frac{d}{ds} \psi_s^{-1}(f_{\Omega_s})|_{s=0}$$

In the case of a volume-preserving transformation, that is when $\operatorname{div} V = 0$ and $\gamma_s = 1$, the mapping ψ_s^{-1} reduces to an intrinsic transformation widely used in differential geometry (see for example [2]). The shape derivative of f is then simply deduced from the Piola derivative by the formula

$$f'_\Omega = \dot{f}_\Omega^P - [V(0), f_\Omega]$$

where $[\cdot,\cdot]$ are the Lie brackets.

2.3 Transported Equation

We prove in this section the part of the theorem 1 dedicated to the existence and uniqueness of $s \mapsto (u_s, p_s)$. We introduce the operators

$$A^s = \psi_s^\star \circ A \circ \psi_s \ \text{ and } \ B^s = \psi_s^\star \circ B \circ (\psi_s, \psi_s) \tag{14}$$

and we call *Transported Navier-Stokes Equations* the system

$$\begin{aligned} \nu A^s u + B^s(u,u) &= \psi_s^\star(\pi(f_{\Omega_s})) \\ u|_\Gamma &= \psi_s^\star(g_{\Gamma_s}) \end{aligned} \tag{15}$$

From the very definition of the operators, a field $u \in H^1(\Omega; \mathbb{R}^3)$ is a solution of the transported Navier-Stokes Equations if and only if $\psi_s(u)$ is a solution of the Navier-Stokes Equations in Ω_s. This is the key property for the analysis of $s \mapsto u_s$ which is made first. The regularity with respect of the shape of the pressure is deduced as a consequence in the following section.

2.3.1 Regularity of the velocity with respect to the shape

The results concerning $s \mapsto u_s$ are consequences of the implicit function theorem applied to the mapping Ψ defined by

$$\Psi(s,u) = \left(\nu A^s u + B^s(u,u) - \psi_s^\star(\pi(f_{\Omega_s}))\,,\, u - \psi_s^\star(g_{\Gamma_s})\right) \tag{16}$$

The implicit function theorem is in fact applied twice: a first time to obtain the continuity in H^{k+1} and then to get the continuous differentiability in H^k only. The functional spaces involved in the theorem are

$$\mathcal{F}_k = \left\{u \in H^k(\Omega;\mathbb{R}^3),\, \operatorname{div} u = 0\right\},\ \mathcal{G}_k = \left\{f \in \pi\left(H^{k-2}(\Omega;\mathbb{R}^3)\right)\right\} \tag{17}$$

this latter space being endowed by the norm induced by $H^{k-2}(\Omega;\mathbb{R}^3)$, and

$$\mathcal{H}_k = \left\{g \in H^{k-1/2}(\Gamma;\mathbb{R}^3),\, \int_\Gamma \langle g, n\rangle \, d\mathcal{H}^2 = 0\right\} \tag{18}$$

Let (u_0, p_0) be the initial solution chosen of the Navier-Stokes Equations in the initial geometry. We have to check the following properties for Ψ:

(i) The continuity of $\Psi : \mathbb{R} \times \mathcal{F}_{k+1} \to \mathcal{G}_{k+1} \times \mathcal{H}_{k+1}$, the existence of $\partial_u \Psi(s, u)$ anywhere and its continuity in in $(0, u_0)$. The continuous differentiability of Ψ as a mapping from $\mathbb{R} \times \mathcal{F}_k$ to $\mathcal{G}_k \times \mathcal{H}_k$.

(ii) The initial solution u_0 belongs to $\mathcal{F}_{k+1}$ and the mapping $\partial_u \Psi(0, u_0)$ is an isomorphism from $\mathcal{F}_k$ to $\mathcal{G}_k \times \mathcal{H}_k$ as well as from $\mathcal{F}_{k+1}$ to $\mathcal{G}_{k+1} \times \mathcal{H}_{k+1}$.

The properties in (i) are consequences of the explicit expression of the transported operators. Simple calculations show that

$$\psi_s^\star \Delta \psi_s(u) = \gamma_s^{-1} J_s^* \cdot \mathrm{div}(D(\gamma_s^{-1} J_s u) \gamma_s J_s^{-1} (J_s^{-1})^*) \tag{19}$$

$$\psi_s^\star((\psi_s(u) \cdot \nabla) \psi_s(v)) = \gamma_s^{-1} J_s^* (u \cdot \nabla)(\gamma_s^{-1} J_s v) \tag{20}$$

The desired regularity is obtained from the properties of the trilinear form $(u, v, w) \mapsto \int_\Omega uvw \, dx$ (see [6], [3]). The same properties also yield with a bootstrap method the properties (ii) as the existence is already known for the solution of the Navier-Stokes Equations and also for the linearized equation from the assumption $\mathcal{A}_2$. Notice that the only thing that prevents Ψ to be continuously differentiable from $\mathbb{R} \times \mathcal{F}_{k+1}$ to $\mathcal{G}_{k+1} \times \mathcal{H}_{k+1}$ is the fact that the data f and g are not regular enough.

As a consequence of (i) and (ii), the implicit function theorem asserts that there is a neighbourhood $I \subset \mathbb{R}$ of 0, a unique family $(u_s)_{s \in I}$ of solutions of the Navier-Stokes Equations in Ω_s such that $s \mapsto \psi_s^{-1}(u_s) \in \mathcal{C}^0(I; H^{k+1}(\Omega; \mathbb{R}^3))$ and $s \mapsto \psi_s^{-1}(u_s) \in \mathcal{C}^1(I; H^k(\Omega; \mathbb{R}^3))$. From the definition of ψ_s we deduce that the same regularity is obtained for the mapping $s \mapsto \phi_s^{-1}(u_s)$.

2.3.2 Regularity of the pressure

The result of the theorem 1 concerning the pressure is obtained by the same transport methods but applied on the classical form of the Navier-Stokes Equations, without using the projector π. We notice that $\psi_s^\star(\nabla p_s) = \nabla \phi_s^{-1}(p_s)$ and therefore that any solution p_s of the Navier-Stokes Equations in Ω_s is subject to

$$\nabla \phi_s^{-1}(p_s) = \nu \cdot \psi_s^\star \Delta u_s - \psi_s^\star((u_s \cdot \nabla) u_s) + \psi_s^\star(f_{\Omega_s})$$

That equation and the results of the previous section yield the desired regularity of $s \mapsto \phi_s^{-1}(p_s)$.

2.4 Shape Derivative and Linearized Equation

In order to establish the equation satisfied by the shape derivatives of u and p, we may use extensions of these shape-dependent mappings. Thanks to the regularity of $s \mapsto u_s$ and $s \mapsto p_s$, there exist two mappings $U \in \mathcal{C}^1(\mathbb{R}; H^1(D;\mathbb{R}^3))$ and $P \in \mathcal{C}^1(\mathbb{R}; L^2(D;\mathbb{R}))$ such that for s small enough, we have $u_{\Omega_s} = U(s)|_{\Omega_s}$ and $p_{\Omega_s} = P(s)|_{\Omega_s}$. Then, the shape derivatives are given by $u'_\Omega = \partial_s U(0)$ and $p'_\Omega = \partial_s P(0)$.

However, this method cannot be applied directly to the right-hand side f for the minimal regularity ($k = 1$). But the shape derivative of f may still be characterized weakly by the following lemma.

Lemma 2 *Let f be a shape-dependent mapping which is $\mathcal{C}^0$ w.r. to the shape in $L^2(\Omega;\mathbb{R}^3)$ and $\mathcal{C}^1$ w.r. to the shape in $H^{-1}(\Omega;\mathbb{R}^3)$. Then for any $\varphi \in \mathcal{D}(\Omega;\mathbb{R}^3)$, the function*

$$s \mapsto \int_{\mathbb{R}^3} \langle f_{\Omega_s}, \varphi \rangle_{\mathbb{R}^3} \, dx$$

is differentiable at $s = 0$ and its derivative is given by

$$\frac{\partial}{\partial s}\left(\int_{\mathbb{R}^3} \langle f_{\Omega_s}, \varphi \rangle_{\mathbb{R}^3} \, dx \right)\Big|_{s=0} = \langle f'_\Omega, \varphi \rangle_{H^{-1}\times H^1_0} \tag{21}$$

Proof – From the definition of the mappings ϕ_s, we deduce that

$$\int_{\mathbb{R}^3} \langle f_{\Omega_s}, \varphi \rangle_{\mathbb{R}^3} \, dx = \left\langle \gamma_s^{-1} \phi_s^\star(f_{\Omega_s}), \gamma_s \varphi \circ T_s \right\rangle_{H^{-1}\times H^1_0}$$

Both arguments of the duality brackets are strongly differentiable. That proves the existence of the derivative. By definition of the material derivative, we have $\dot{f}_\Omega = \frac{\partial}{\partial s} \gamma_s^{-1} \phi_s^\star(f_{\Omega_s})|_{s=0}$ and clearly, $\frac{\partial}{\partial s} \gamma_s \varphi \circ T_s|_{s=0} = \operatorname{div} V(0) \cdot \varphi + D\varphi \cdot V(0) = \operatorname{div}(\varphi \otimes V(0))$. Consequently, we have

$$\begin{aligned} \frac{\partial}{\partial s}\left(\int_{\mathbb{R}^3} \langle f_{\Omega_s}, \varphi \rangle_{\mathbb{R}^3} \, dx \right)\Big|_{s=0} &= \left\langle \dot{f}_\Omega, \varphi \right\rangle_{H^{-1}\times H^1_0} \\ &\quad + \int_{\mathbb{R}^3} \langle f_\Omega, \operatorname{div}(\varphi \otimes V(0)) \rangle_{\mathbb{R}^3} \, dx \end{aligned}$$

On the other hand, as

$$\begin{aligned} \int_{\mathbb{R}^3} \langle f_\Omega, \operatorname{div}(\varphi \otimes V(0)) \rangle_{\mathbb{R}^3} \, dx &= -\langle Df_\Omega, \varphi \otimes V(0) \rangle_{H^{-1}\times H^1_0} \\ &= -\langle Df_\Omega \cdot V(0), \varphi \rangle_{H^{-1}\times H^1_0} \end{aligned}$$

using the definition of f'_Ω, we obtain the equality (21). ■

Let $\varphi \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ such that $\mathrm{Supp}(\varphi) \subset \Omega$. For small values of s, the support of φ is also included in Ω_s and therefore

$$\int_D \nu \partial_x U(s) \cdot\cdot\, \partial_x \varphi + \langle \partial_x U(s) \cdot U(s), \varphi \rangle_{\mathbb{R}^3} \, dx$$
$$= \int_{\mathbb{R}^3} P(s) \cdot \operatorname{div} \varphi + \langle f_{\Omega_s}, \varphi \rangle_{\mathbb{R}^3} \, dx$$

The differentiation of this equation with respect to s gives

$$\int_D \nu \partial_x \partial_s U(0) \cdot\cdot\, \partial_x \varphi + \langle [\partial_x \partial_s U(0)]\, U(0) + [\partial_x U(0)] \partial_s U(0), \varphi \rangle_{\mathbb{R}^3} \, dx$$
$$= \int_D \partial_s P(0) \cdot \operatorname{div} \varphi \, dx + \langle f'_\Omega, \varphi \rangle_{H^{-1} \times H_0^1}$$

which implies that u' and p' are solutions of (9). The equation $\operatorname{div} u' = 0$ is a direct consequence of $\operatorname{div} u_s = 0$ and of the existence of the extension $s \mapsto U(s)$.

The boundary condition is proved as follows. As $u|_\Gamma = g_\Gamma$, $u'_\Gamma = g'_\Gamma$. On the other hand

$$u'_\Gamma = u'|_\Gamma + \frac{\partial u}{\partial n} \langle V(0), n \rangle$$

therefore, we obtain the desired Dirichlet boundary condition.

3 Lagrangian Functionals

In order to investigate the sensitivity of shape functionals of lagrangian type, we state some sensitivity results for the solutions of the ordinary differential equations. The analysis is done for time-dependent mappings with values in $\mathcal{C}^k$ spaces rather than in the classical framework of two-variable mappings (time and space).

Before this, we briefly analyze the shape differentiability of a simple distributed shape functional. The corresponding results are needed as tools for the analysis of the lagrangian functionals.

3.1 Shape Differentiability of a distributed functional

From now on, we assume that the data considered in the theorem 1 satisfy the assumptions $\mathcal{A}_1$ with $k = 1$ and $\mathcal{A}_2$. Let $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^1$ function whose support is compactly included in the reference set $\Omega_0 \subset D$. We consider the (possibly multi-valued) shape functional J, given by

$$J(\Omega) = \int_{\Omega} \langle \rho, u \rangle_{\mathbb{R}^3} \; dx \tag{22}$$

where u is a solution of the system of equations (1). For any $V \in \mathcal{V}$ and any solution u_0 of the Navier-Stokes Equations in Ω_0, the theorem 1 provides a family $(u_s)_{s \in I}$ that determines uniquely a value of the shape functional in Ω_s, value still denoted $J(\Omega_s)$. As a consequence of the regularity of $(u_s)_{s \in I}$ given in the theorem 1 and by the Reynolds formula (see [10]), the eulerian derivative of J in the direction $V \in \mathcal{V}$, defined as $\mathsf{d}J(\Omega; V) = \frac{d}{ds} J(\Omega_s)|_{s=0}$, exists and is given by

$$\mathsf{d}J(\Omega; V) = \int_{\Omega} \langle \rho, u' \rangle \; dx \tag{23}$$

where u' is the solution of (9). The introduction of the corresponding adjoint system allows further simplifications:

$$\begin{array}{rcll} -\nu \Delta \eta - D\eta \cdot u_0 + [Du_0]^* \cdot \eta + \nabla \pi & = & \rho & \text{in } \Omega_0 \\ \operatorname{div} \eta & = & 0 & \text{in } \Omega_0 \\ \eta & = & 0 & \text{on } \Gamma_0 \end{array} \tag{24}$$

The assumption $\mathcal{A}_2$ ensures the existence and uniqueness of the solution η. The regularity of u, deduced from the assumption $\mathcal{A}_1$ and a bootstrapping method shows that $\eta \in H^2(\Omega; \mathbb{R}^3)$. Therefore, we have

$$\begin{array}{rcl} \mathsf{d}J(\Omega; V) & = & \displaystyle\int_{\Omega} \langle -\nu \Delta \eta - D\eta \cdot u_0 + [Du_0]^* \cdot \eta + \nabla \pi, u' \rangle \;\; dx \\ & = & \displaystyle\int_{\Omega} \langle \eta, f'_{\Omega} \rangle \; dx + \int_{\Gamma} \left\langle -\nu \frac{\partial \eta}{\partial n} + \pi n, u' \right\rangle \; dx \\ & = & \displaystyle\int_{\Omega} \langle \eta, f'_{\Omega} \rangle \; dx + \int_{\Gamma} \left\langle -\nu \frac{\partial \eta}{\partial n} + \pi n \; , \; g'_{\Gamma} - \frac{\partial u}{\partial n} \langle V(0), n \rangle \right\rangle \; dx \end{array}$$

This expression may be rewritten in terms of *stress tensor*. We associate to $u \in H^1(\Omega; \mathbb{R}^3)$ and $p \in L^2(\Omega; \mathbb{R})$ the matrix

$$\sigma(u, p) = pI - \nu(Du + Du^*) \tag{25}$$

Some additional information on the structure of the normal component of $\sigma(u, p)$ on the boundary may be obtained. Let D_τ and div_τ be respectively the tangential Jacobian and the tangential divergence operators. We state the

Lemma 3 *Let u be a divergence-free field of $H^2(\Omega;\mathbb{R}^3)$ such that $u|_\Gamma = h$. Then, we have*

$$Du^* n = [D_\tau h^* - (\mathrm{div}_\tau h) I]\, n \tag{26}$$

Proof – Using the decomposition of the operators D and div on the boundary, we obtain

$$Du = D_\tau h + \frac{\partial u}{\partial n} \otimes n \ \text{ and } \ \mathrm{div}\, u = \mathrm{div}_\tau h + \left\langle \frac{\partial u}{\partial n}, n \right\rangle = 0 \tag{27}$$

Consequently, we have

$$\begin{aligned} [Du^*]n &= \left[D_\tau h^* + n \otimes \frac{\partial u}{\partial n} \right] n \\ &= [D_\tau h^*]n + \left\langle \frac{\partial u}{\partial n}, n \right\rangle n \\ &= [D_\tau h^* - (\mathrm{div}_\tau h) I]\, n \end{aligned}$$

■

As a direct consequence of that lemma, the derivative of J is equal to

$$\mathsf{d}J(\Omega;V) = \int_\Omega \langle \eta, f'_\Omega \rangle \, dx + \int_\Gamma \left\langle \sigma(\eta,\pi)n \ , \ g'_\Gamma - \frac{\partial u}{\partial n} \langle V(0), n \rangle \right\rangle d\mathcal{H}^2 \tag{28}$$

3.2 Shape derivation of the velocity flow

3.2.1 Sensitivity in the ODEs

In this section, K is a compact set of $\mathbb{R}^n$ such that $\overline{D} \subset K$ and $T > 0$. We define the set $\mathcal{H} = \{f \in \mathcal{C}^0(\mathbb{R}^n;\mathbb{R}^n), \mathrm{Supp}(f) \subset K\}$ and denote $\|\cdot\|_k$ the usual norm on the set $\mathcal{C}^k(\overline{A};F)$. If F is itself of the form $F = \mathcal{C}^l(\overline{B};G)$, we also use the notation $\|\cdot\|_{k,l}$.

Lemma 4 *The mappings*

$$\begin{array}{ccc} \mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}))^2 & \to & \mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R})) \\ (f,g) & \mapsto & fg \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C}^0([-T;T];\mathcal{H}) \times \mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}^n)) & \to & \mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}^n)) \\ (f,g) & \mapsto & [t \mapsto f(t) \circ g(t)] \end{array}$$

are continuous.

Proof –

i) clear (remember that $\mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R})) \simeq \mathcal{C}^0([-T;T]\times\overline{D};\mathbb{R}))$.

ii) for any f, f' in $\mathcal{C}^0([-T;T];\mathcal{H})$, and g, g' in $\mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}^n))$, we have

$$\|f'\circ g' - f\circ g\|_{0,0} \leq \|f'-f\|_{0,0} + \|f\circ g' - f\circ g\|_{0,0}$$

It is therefore enough to show the uniform equicontinuity of the family $(f(t))_{t\in[-T;T]}$. It is proved as follows: For any $t\in[-T;T]$, and any $\varepsilon>0$, there is a $\eta>0$ such that $\|x-y\|<\eta$ implies $\|f(t)(x)-f(t)(y)\|<\varepsilon/3$ (by uniform continuity of $f(t)$). Let $\tau>0$ be such that for any $t'\in[-T;T]\cap]t-\tau;t+\tau[$, $\|f(t')-f(t)\|_0<\varepsilon/3$. Then, for any x and y such that $\|x-y\|<\eta$, $\|f(t')(x)-f(t')(y)\|<\varepsilon$. By compactness of $[-T;T]$, the result is proved. ■

Assume that D is of class $\mathcal{C}^2$. Using an extension operator, we may identify the velocity space $\mathcal{V}_2$ with a linear subspace of the set of continuous mappings from $[-T;T]$ to $\{f\in\mathcal{C}^2(\mathbb{R}^n;\mathbb{R}^n),\ \mathrm{Supp}(f)\subset K\}$. For any mapping $\xi\in\mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and $v\in\mathcal{V}_2$, we set

$$H(\xi,v) = [t\mapsto v(t)\circ\xi(t)] \tag{29}$$

and we state the

Lemma 5 *The expression (29) defines a $\mathcal{C}^1$ mapping*

$$H:\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))\times\mathcal{V}_2\to\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$$

whose differential is given by:

$$\begin{aligned} DH(\xi,v)\cdot(\zeta,W) &= \partial_\xi H(\xi,v)\cdot\zeta + \partial_v H(\xi,v)\cdot W & (30)\\ &= \Big[t\mapsto [D_x v(t)\circ\xi(t)]\,\zeta(t) + W(t)\circ\xi(t)\Big] & (31)\end{aligned}$$

Proof – The mapping H is well-defined: indeed for any $t\in[-T;T]$, ξ belongs to $\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and $v\in\mathcal{V}_2$, the function $H(\xi,v)(t)$ is differentiable and $D_x[H(\xi,v)(t)] = [D_x v(t)]\circ\xi(t)\cdot D_x\xi(t)$. The lemma 4 yields that $D_x H(\xi,v)$ belongs to $\mathcal{C}^0([-T;T];\mathcal{C}^0(\overline{D};\mathbb{R}^{n\times n}))$ and therefore $H(\xi,v)\in\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^{n\times n}))$.

• Existence and continuity of $\partial_\xi H$: let $\mathcal{A}_{\xi,v}$ be the linear operator $\zeta\mapsto\big[t\mapsto A_{\xi,v}(t)\zeta(t)\big]$ with $A_{\xi,v}(t) = D_x v(t)\circ\xi(t)$. The function $A_{\xi,v}(t)$ is differentiable for any $t\in[-T;T]$ and $D_x A_{\xi,v}(t) = [D_x^2 v(t)\circ\xi(t)]\,[D_x\xi(t)]$. Therefore $A_{\xi,v}$ belongs to $\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and depends continuously on (ξ,v) (lemma 4). Consequently, $\mathcal{A}_{\xi,v}$ is a well-defined and continuous mapping from $\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ to $\mathcal{C}^0([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and the mapping $(\xi,v)\mapsto\mathcal{A}_{\xi,v}$ is continuous.

Let us prove the existence of $\partial_\xi H$ and the equality $\partial_\xi H(\xi, v) = \mathcal{A}_{\xi,v}$. We set

$$\Sigma = v \circ (\xi + \zeta) - v \circ \xi - D_x v \circ \xi \cdot \zeta \tag{32}$$

Straightforward calculations show that for any $i \in \{1, ..., n\}$, $D_x \Sigma_i = D_x v_i \circ (\xi + \zeta) \cdot D_x(\xi + \zeta) - D_x v_i \circ \xi \cdot D_x \xi - \zeta^* \cdot [D_x^2 v_i \circ \xi] \cdot D_x \xi - D_x v_i \circ \xi \cdot D_x \zeta$ and therefore,

$$D_x \Sigma_i = K_i^* \cdot D_x \xi + L_i^* \cdot D_x \zeta \tag{33}$$

$$\text{with } \begin{cases} K_i = L_i - D_x^2 v_i \circ \xi \cdot \zeta \\ L_i = \nabla_x v_i \circ (\xi + \zeta) - \nabla_x v_i \circ \xi \end{cases}$$

We use the following intermediate result: we set $\varphi = v$ or $\varphi = \nabla_x v_i$ for a $i \in \{1, ..., n\}$. Then, there is a function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$, independent of (t, x), with $\lim_{h\to 0} \varepsilon(h) = 0$ and such that for any $t \in [-T; T]$ and $x \in \mathbb{R}^n$, the inequality

$$\|\varphi(t)(x) - \varphi(t)(x + h) - D_x \varphi(t)(x) \cdot h\| = \varepsilon(h) \cdot \|h\| \tag{34}$$

holds. To prove this, we define $\psi(t, x, h) = \varphi(t)(x + h) - D_x \varphi(t)(x) \cdot (x + h)$ and notice that $D_h \psi(t, x, h) = D_x \varphi(t)(x+h) - D_x \varphi(t)(x)$. From the uniform equicontinuity of $(D_x \varphi(t))_{t\in[-T;T]}$ (see again the point *ii)* in the proof of the lemma 4), we deduce that there is a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{h\to 0} \lambda(h) = 0$ and $\|D_h \psi(t, x, h)\| \leq \lambda(h)$. Therefore

$$\begin{aligned} \varphi(t)(x) - \varphi(t)(x + h) - D_x \varphi(t)(x) \cdot h &= \psi(t, x, 0) - \psi(t, x, h) \\ &= \int_0^1 [D_h \psi(t, x, \theta h)] \cdot h \, d\theta \end{aligned}$$

and consequently, the equation (34) holds with $\varepsilon(h) = \sup_{\delta\in[0,h]} \lambda(\delta)$.

From this result, we deduce the inequalities $\|\Sigma\|_{0,0} \leq \varepsilon(\|\zeta\|_{0,1}) \cdot \|\zeta\|_{0,1}$, $\|K_i\|_{0,0} \leq \varepsilon(\|\zeta\|_{0,1}) \cdot \|\zeta\|_{0,1}$ and $\|L_i\|_{0,0} \leq (\varepsilon(\|\zeta\|_{0,1}) + \|v\|_{\mathcal{V}_2}) \cdot \|\zeta\|_{0,1}$. Finally, using the expressions (32) and (33), we end up with

$$\|\Sigma\|_{0,0} + \|D_x \Sigma\|_{0,0} \leq \eta(\|\zeta\|_{0,1}) \cdot \|\zeta\|_{0,1} \;\text{ with }\; \lim_{h\to 0} \eta(h) = 0 \tag{35}$$

which proves the desired differentiability result.

• Existence and continuity of $\partial_v H$: the existence and expression of $\partial_v H(\xi, v)$ is clear: the mapping $[v \mapsto H(\xi, v)]$ is linear continuous and therefore $\partial_v H(\xi, v) \cdot W = H(\xi, W)$. The continuity of this partial derivative is a direct consequence of the lemma 4. ■

3.2.2 Differentiability of the flow

Let v be a vector field of $\mathcal{V}_1$. The corresponding flow is hereafter denoted T_v or $T(v)$ to emphasize its dependence on v; it is the solution of the initial-value problem

$$\begin{array}{l} \partial_t T_v(t) = v(t) \circ T_v(t) \\ T_v(0) = I \end{array}$$

We know that $T(v)$ belongs to $\mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ (see [10]). The following proposition characterizes the regularity of the correspondence $v \mapsto T(v)$ under a stronger assumption on the regularity of the variable v.

Proposition 2 *The mapping*

$$\begin{array}{ccc} \mathcal{V}_2 & \to & \mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n)) \\ v & \mapsto & T(v) \end{array}$$

is continuously differentiable. For any $W \in \mathcal{V}_2$, $\zeta = \partial_v T(v) \cdot W$ *is the solution of*

$$\begin{array}{l} \partial_t \zeta(t) = [D_x v(t) \circ T_v(t)] \cdot \zeta(t) + W(t) \circ T_v(t) \\ \zeta(0) = 0 \end{array} \tag{36}$$

Proof – For any $\xi \in \mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$ and $v \in \mathcal{V}_2$, we set

$$F(\xi, v) = \xi - \left[t \mapsto I + \int_0^t v(\tau) \circ \xi(\tau)\, d\tau \right] \tag{37}$$

The unique solution of $F(\xi, v) = 0$ is $\xi = T(v)$. Moreover, from the lemma 5 we deduce that $(\xi, v) \mapsto F(\xi, v)$ is a $\mathcal{C}^1$ mapping from $\mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n)) \times \mathcal{V}_2$ to $\mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$. Its differential is given by:

$$DF(\xi, v) \cdot (\zeta, W) = \zeta - \left[t \mapsto \int_0^t ([D_x v(\tau) \circ \xi(\tau)] \cdot \zeta(\tau) + W(\tau) \circ \xi(\tau))\, d\tau \right]$$

Therefore, $\partial_\xi F(\xi, v)$ is an isomorphism: for any $\phi \in \mathcal{C}^1([-T;T];\mathcal{C}^1(\overline{D};\mathbb{R}^n))$, we set $\psi = \partial_t \phi$. The mapping ϕ satisfies $\partial_\xi F(\xi, v) \cdot \zeta = \phi$ iff it is the solution of

$$\begin{array}{l} \partial_t \zeta(t) = [D_x v(\tau) \circ \xi(t)] \cdot \zeta(t) + \psi(t) \\ \zeta(0) = \phi(0) \end{array}$$

which is given by

$$\zeta(t) = U(0, t)\phi(0) + \int_0^t U(t, \tau)\psi(\tau)\, d\tau$$

with $U(s,t) = \exp\left(\int_s^t D_x v(\tau) \circ \xi(\tau)\, d\tau\right)$.

The regularity of $v \mapsto T(v)$ is a consequence of the implicit function theorem. The expression $\zeta = \partial_v T(v) \cdot W$ is solution of $\partial_\xi F(\xi, v) \cdot \zeta = -\partial_v F(\xi, v) \cdot W$ or equivalently, of the system (36). ■

Now, we investigate the regularity with respect to the shape of a flow based on a shape-dependent mapping v. We assume that this mapping satisfies the following properties:

$$v \text{ is } \mathcal{C}^1 \text{ w.r. to the shape in } \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3) \text{ and } \mathcal{C}^0 \text{ w.r. to the shape in } \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^3). \text{ Moreover, for any admissible } \Omega,\ \langle v_\Omega, n_\Omega \rangle = 0 \tag{38}$$

We describe the regularity of the corresponding shape-dependent flow $T(v)$ by the regularity of its extensions to $\overline{D}$.

Proposition 3 *Let $\Omega \subset D$ be an open bounded set of class $\mathcal{C}^3$ such that either $\Omega \subset\subset D$ or $\overline{D} - \overline{\Omega}$ is compactly included in D. Then, for any $V \in \mathcal{V} \subset \mathcal{V}_3$*

(i) *There is a continuously differentiable mapping $s \mapsto R_s$ with values in $\mathcal{C}^1([-T;T]; \mathcal{C}^1(\overline{D}; \mathbb{R}^3))$ such that for any $s \in \mathbb{R}$ and $t \in [-T;T]$, $R_s(t)$ is an extension of $T(v_{\Omega_s})(t)$ to $\overline{D}$.*

(ii) *The restriction S of $\partial_s R_s|_{s=0}$ to Ω is the solution of*

$$\begin{aligned} &\dot{S}(t) = v' \circ T_v(t) + [D_x v \circ T_v(t)] S(t) \\ &S(0) = 0 \end{aligned} \tag{39}$$

Proof – The assumption (38) yields the existence of a mapping w that belongs to $\mathcal{C}^1(\mathbb{R}, \mathcal{C}^2(\overline{D}; \mathbb{R}^3))$ such that for any $s \in \mathbb{R}$, $w(s)|_{\Omega_s} = v_{\Omega_s}$. The shape derivative v'_Ω is equal to $\partial_s w(0)|_\Omega$. We may moreover assume that $w(s) \in \mathcal{V}_2$. Let R_s be the flow associated to $w(s)$, solution of

$$\begin{aligned} \partial_t R_s(t) &= w(s) \circ R_s(t) \\ R_s(0) &= I \end{aligned} \tag{40}$$

Obviously, we have $R_s(t)|_{\Omega_s} = T_{v_{\Omega_s}}(t)$ and the proposition 2 yields $s \mapsto R_s \in \mathcal{C}^1(\mathbb{R}; \mathcal{C}^1([-T;T]; \mathcal{C}^1(\overline{D}; \mathbb{R}^3)))$ and $\bar{S} = \partial_s R_s|_{s=0}$ satisfies

$$\begin{aligned} &\partial_t \bar{S}(t) = \partial_s w(0)(t) \circ T_{w(0)}(t) + [D_x w(0)(t) \circ T_{w(0)}(t)] \cdot \bar{S}(t) \\ &\bar{S}(0) = 0 \end{aligned} \tag{41}$$

which yields (39). ■

3.3 Shape Gradient of a Lagrangian Functional

Let $\rho : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^\infty$ function such that $\forall\, t \in [0,T],\ \mathrm{Supp}(\rho(t,\cdot)) \subset \Omega$. We consider the shape functional J, defined by

$$J(\Omega) = \iint_{[0,T]\times D} \langle \rho(t,x), T_u(t)(x)\rangle \; dtdx \tag{42}$$

where u is solution of the Navier-Stokes Equations in Ω. We assume that the assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ are satisfied with $k = 4$ and moreover that $\langle g, n_\Omega\rangle_{\mathbb{R}^3} = 0$ and either $\Omega \subset\subset D$ or $\overline{D} - \overline{\Omega} \subset\subset D$. Under these assumptions, the theorem 1 and the Sobolev injections imply that u satisfies the assumptions (38) and therefore the proposition 3 holds for the flow T_u. Consequently, we have the

Proposition 4 *The eulerian derivative of J exists and is given by*

$$\mathsf{d}J(\Omega;V) = \int_\Omega \langle \eta, f'_\Omega\rangle \; dx + \int_\Gamma \left\langle \sigma(\eta,\pi)n\,,\, g'_\Gamma - \frac{\partial u}{\partial n}\,\langle V(0), n\rangle \right\rangle \, d\mathcal{H}^2 \tag{43}$$

where

$$\begin{array}{l} \dot{Q}(t) = -[Du^*]Q(t) - [DQ(t)]u - \rho(t,\cdot)\circ [T_u(t)]^{-1} \\ Q(T) = 0 \end{array} \tag{44}$$

and

$$\begin{array}{rcll} -\nu\Delta\eta - D\eta\cdot u + [Du]^*\cdot\eta + \nabla\pi &=& \int_0^T Q(t)\,dt & in\ \Omega \\ \mathrm{div}\,\eta &=& 0 & in\ \Omega \\ \eta &=& 0 & on\ \Gamma \end{array} \tag{45}$$

Proof – We introduce the adjoint equation where the adjoint state is P:

$$\begin{array}{l} \dot{P}(t) = -\rho(t,\cdot) - [Du\circ T_u(t)]^*\cdot P(t) \\ P(T) = 0 \end{array} \tag{46}$$

Then, the eulerian derivative of J exists and thanks to the equation (39) we obtain:

$$\begin{array}{rcl} \mathsf{d}J(\Omega;V) &=& \displaystyle\iint_{[0,T]\times\Omega} \langle \rho(t,\cdot), S(t)\rangle \; dtdx \\ &=& \displaystyle\iint_{[0,T]\times\Omega} \left\langle -\dot{P}(t) - [Du\circ T_u(t)]^*\cdot P(t)\,,\, S(t)\right\rangle \, dtdx \\ &=& \displaystyle\iint_{[0,T]\times\Omega} \left\langle P(t)\,,\, \dot{S}(t) - [Du\circ T_u(t)]S(t)\right\rangle \, dtdx \\ &=& \displaystyle\iint_{[0,T]\times\Omega} \langle P(t)\,,\, u'\circ T_u(t)\rangle \; dtdx \end{array}$$

This expression may be further simplified by the introduction of $Q(t)$

$$Q(t) = P(t) \circ [T_u(t)]^{-1} \tag{47}$$

which is the solution of the PDE problem (44): the differentiation with respect to t of the equation $Q(t) \circ T_u(t) = P(t)$ gives $\dot{Q}(t) \circ T_u(t) + [DQ(t) \circ T_u(t)] \cdot \dot{T}_u(t) = \dot{P}(t)$ and therefore

$$\dot{P}(t) \circ [T_t^u]^{-1} = \dot{Q}(t) + DQ(t) \cdot (\dot{T}_t^u \circ [T_t^u]^{-1})$$

The equation (44) simply results from the substitution of this equality in (46). Then, as $\operatorname{div} u = 0$, $|\det DT_t^u| = 1$ and by a change of variable, we find that

$$\begin{aligned}
\mathsf{d}J(\Omega; V) &= \iint_{[0,T]\times\Omega} \langle P(t)\,,\, u' \circ T_t^u \rangle \; dtdx \\
&= \iint_{[0,T]\times\Omega} \langle Q(t)\,,\, u' \rangle \; dtdx \\
&= \int_\Omega \left\langle \int_0^T Q(t)\, dt\,,\, u' \right\rangle \, dx
\end{aligned}$$

so as a consequence of the section 3.1, we finally obtain the expression of the eulerian derivative given in the proposition 4. ■

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