C1223 – Complex Analysis
Final Examination

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Problem R

Preamble. Let Ω be an open subset of $\mathbb{C}$. For any $r > 0$, we denote Ω_r the set of points of $\mathbb{C}$ whose distance to the complement of Ω is larger than r:

$$\Omega_r = \{z \in \mathbb{C} \mid d(z, \mathbb{C} \setminus \Omega) > r\}.$$

1. Show that Ω_r is an open subset of Ω and that $\Omega = \cup_{r>0} \Omega_r$.
2. Assume that Ω is connected. Is Ω_r necessarily connected? (Hint: consider for example $\Omega = \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < |\mathrm{Re}\, z| + 1\}$ and $r = 1$).
3. Show that if $z \in \mathbb{C} \setminus \Omega_r$, there is a $w \in \mathbb{C} \setminus \Omega$ such that the segment $[w, z]$ is included in $\mathbb{C} \setminus \Omega_r$. Deduce from this property that if Ω is simply connected, then Ω_r is also simply connected. Is the converse true?
4. Show that if Ω is bounded and simply connected, $\mathbb{C} \setminus \Omega$ is connected. (Hint: assume that Ω is bounded but that $\mathbb{C} \setminus \Omega$ is disconnected, then introduce a suitable dilation of this complement).

From now on, Ω is a bounded open subset of $\mathbb{C}$. Let $\mathcal{F}$ be a class of holomorphic functions defined on Ω (or a superset of Ω). A holomorphic function $f : \Omega \to \mathbb{C}$ has *uniform approximations* in $\mathcal{F}$ if

$$\forall\, \epsilon > 0, \exists\, \hat{f} \in \mathcal{F},\ \forall\, z \in \Omega,\ |f(z) - \hat{f}(z)| \leq \epsilon.$$

5. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and let $a \in \mathbb{C} \setminus \Omega$. Assume that f has uniform approximations in the class of functions defined and holomorphic on $\mathbb{C} \setminus \{a\}$. Show that if $|a|$ is large enough, f has uniform approximations among polynomials.
6. Show that for any non-empty bounded open subset Ω of $\mathbb{C}$, there is a holomorphic function $f : \Omega \to \mathbb{C}$ which doesn't have uniform approximations among polynomials (Hint: consider $z \mapsto 1/(z - a)$ for some suitable choice of a).

A holomorphic function $f : \Omega \to \mathbb{C}$ has *locally uniform approximations* in $\mathcal{F}$ if for any $r > 0$, its restriction to any Ω_r has uniform approximations in $\mathcal{F}$:

$$\forall\, \epsilon > 0, \forall\, r > 0, \exists\, \hat{f} \in \mathcal{F},\ \forall\, z \in \Omega_r,\ |f(z) - \hat{f}(z)| \leq \epsilon.$$

7. Show that if Ω is not simply connected, there is a holomorphic function $f : \Omega \to \mathbb{C}$ which has no locally uniform approximations among polynomials (Hint: consider $f : z \mapsto 1/(z-a)$ for some suitable choice of a then compare

$$\int_\gamma f(z)\, dz \quad \text{and} \quad \int_\gamma \hat{f}(z)\, dz$$

   for some suitable closed rectifiable path γ).

From now on, we assume that Ω is simply connected.

Let $f : \Omega \to \mathbb{C}$ and let $r > 0$. We define the function $\chi_r : \mathbb{C} \setminus \Omega \to \{0, 1\}$ by:

- $\chi_r(z) = 1$ if for every $\epsilon > 0$, there is a holomorphic function $\hat{f}_z$ defined on $\mathbb{C} \setminus \{z\}$ such that $|f - \hat{f}_z| \leq \epsilon$ on Ω_r,
- $\chi_r(z) = 0$ otherwise.

8. Show that if some points z and w of $\mathbb{C} \setminus \Omega$ satisfy $\chi_r(z) = 1$ and $|w - z| < r/2$ then $\chi_r(w) = 1$ (Hint: first, prove that the open annulus $A := A(w, r/2, +\infty)$ satisfies $A \subset \mathbb{C} \setminus \{z\}$ and $\overline{\Omega_r} \subset A$).

9. Prove that χ_r is locally constant then show that if $\chi_r(a) = 1$ for some $a \in \mathbb{C} \setminus \Omega$, then $\chi_r(z) = 1$ for every $z \in \mathbb{C} \setminus \Omega$.

10. Assume that $f : \Omega \to \mathbb{C}$ has locally uniform approximations among holomorphic functions defined on $\mathbb{C} \setminus \{a\}$ for some $a \in \mathbb{C} \setminus \Omega$. Show that f has locally uniform approximations among polynomials.

11. Let $\hat{f} : \mathbb{C} \setminus \{a_1, \dots, a_n\} \to \mathbb{C}$ be holomorphic (all the a_k are distincts). Show that there are holomorphic functions $\hat{f}_k : \mathbb{C} \setminus \{a_k\} \to \mathbb{C}$ for $k = 1, \dots, n$ such that

$$\forall\, z \in \mathbb{C} \setminus \{a_1, \dots, a_n\},\ \hat{f}(z) = \hat{f}_1(z) + \cdots + \hat{f}_n(z).$$

    Prove the following corollary: if a function $f : \Omega \to \mathbb{C}$ has locally uniform approximations among holomorphic functions defined on $\mathbb{C} \setminus \{a_1, \dots, a_n\}$ for some $a_1, \dots, a_n \in \mathbb{C} \setminus \Omega$, then f has locally uniform approximations among polynomials.

Problem L

The *ray* with origin $a \in \mathbb{C}$ and direction $u \in \mathbb{C}^*$ is the function[1]

$$\gamma : t \in \mathbb{R}_+ \mapsto a + tu.$$

Let $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function defined on some open subset Ω of $\mathbb{C}$ that contains the image $\gamma(\mathbb{R}_+)$ of the ray γ. The Laplace transform $\mathcal{L}_\gamma[f]$ of f along γ at $s \in \mathbb{C}$ is given by

$$\mathcal{L}_\gamma[f](s) = \int_{\mathbb{R}_+} f(\gamma(t)) e^{-\gamma(t)s} \gamma'(t)\, dt$$

(we consider that this integral is defined when its integrand is summable). This definition generalizes the classic Laplace transform $\mathcal{L}[f]$ since $\mathcal{L}_\gamma[f] = \mathcal{L}[f]$ when $\gamma(t) = t$ (that is when $a = 0$ and $u = 1$).

We assume that there are some $\kappa > 0$ and $\sigma \in \mathbb{R}$ such that

$$\forall\, z \in \gamma(\mathbb{R}_+),\ |f(z)| \leq \kappa e^{\sigma |z|}. \tag{1}$$

1. Show that if $\gamma(t) = a + tu$ and $\mu(t) = a + t(\lambda u)$ for some $\lambda > 0$, then

   $$\mathcal{L}_\mu[f] = \mathcal{L}_\gamma[f]$$

   (Reminder: two functions are equal when the have the same domain of definition and the same values in this shared domain.)

2. Characterize geometrically the set

   $$\Pi(u, \sigma) = \{ s \in \mathbb{C} \mid \mathrm{Re}\,(su) > \sigma |u| \}$$

   and show that $\mathcal{L}_\gamma[f]$ is defined and holomorphic on $\Pi(u, \sigma)$.

3. Let U be an open subset of $\mathbb{C}^*$. We assume that bound (1) is valid for every $u \in U$ (for a given origin a and fixed values of κ and σ). Show that for any $s \in \mathbb{C}$, the set U_s of directions $u \in U$ such that $s \in \Pi(u, \sigma)$ is open and that the function $u \in U_s \mapsto \mathcal{L}_\gamma[f](s)$ is holomorphic (Hint: show that the complex-differentiation under the integral sign theorem is applicable).

4. Show that the derivative of $\mathcal{L}_\gamma[f](s)$ with respect to u is zero (Hint: the result of question 1 may be used).

The exponential integral $E_1(x)$ is defined for $x > 0$ by

$$E_1(x) = \int_x^{+\infty} \frac{e^{-t}}{t} dt.$$

[1] This notation emphasizes that the complex number $\gamma(t)$ depends on t; however it also depends implicitly on some a and u that are usually clear from the context. Feel free to use a more explicit notation if you feel that it is beneficial.

5. Compute the (classic) Laplace transform F of

$$t \in \mathbb{R}_+ \to \frac{1}{t+1}$$

and give a formula for $E_1(x)$ that depends on $F(x)$. Prove that E_1 has a unique holomorphic extension to the open right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$.

From now on, we study the case of

$$f : z \in \mathbb{C} \setminus \{-1\} \mapsto \frac{1}{z+1}$$

with $a = 0$, $\sigma = 0$ and $U = \{u \in \mathbb{C} \mid \mathrm{Re}(u) > 0\}$.

6. Show that there is a $\kappa > 0$ such that (1) is valid for every $u \in U$. Characterize geometrically the set U_s; show that it is non-empty when $s \in \mathbb{C} \setminus \mathbb{R}_-$.

Let $s \in \mathbb{C} \setminus \mathbb{R}_-$. We define

$$G(s) := \mathcal{L}_\gamma[f](s) \text{ if } u \in U_s \text{ and } \gamma : t \in \mathbb{R}_+ \mapsto tu.$$

Note that this definition is *a priori* ambiguous since several $u \in U_s$ exist for a given value of s. For any $u_0 \in \mathbb{C}^*$ and $u_1 \in \mathbb{C}^*$ and for any $\theta \in [0, 1]$, we denote

$$u_\theta = (1 - \theta)u_0 + \theta u_1$$

and whenever $u_\theta \neq 0$, we denote γ_θ the ray of origin $a = 0$ and direction u_θ.

7. Let $s \in \mathbb{C} \setminus \mathbb{R}_-$. Show that if $u_0 \in U_s$ and $u_1 \in U_s$ then for every $\theta \in [0, 1]$, $u_\theta \in U_s$ and that

$$\mathcal{L}_{\gamma_1}[f](s) - \mathcal{L}_{\gamma_0}[f](s) = \int_0^1 \frac{d}{d\theta} \mathcal{L}_{\gamma_\theta}[f](s)\, d\theta.$$

Conclude that the definition of G is unambiguous.

8. We search for a new expression of the difference $\mathcal{L}_{\gamma_1}[f](s) - \mathcal{L}_{\gamma_0}[f](s)$ to build an alternate proof for the conclusion of the previous question.

Let again $s \in \mathbb{C} \setminus \mathbb{R}_-$, $u_0 \in U_s$ and $u_1 \in U_s$; let $r \geq 0$ and γ_0^r and γ_1^r be the paths defined by

$$\gamma_0^r : t \in [0, 1] \mapsto \gamma_0(tr) \text{ and } \gamma_1^r : t \in [0, 1] \mapsto \gamma_1(tr)$$

Show that

$$\mathcal{L}_{\gamma_1}[f](s) - \mathcal{L}_{\gamma_0}[f](s) = \lim_{r \to +\infty} \int_{\mu_r} f(z) e^{-sz}\, dz \text{ where } \mu_r = (\gamma_0^r)^{\leftarrow} | (\gamma_1^r)$$

Conclude again that the definition of G is unambiguous (Hint: "close" the path μ_r and use Cauchy's integral theorem).

9. Prove that E_1 has a unique holomorphic extension to $\mathbb{C} \setminus \mathbb{R}_-$.

Problem R – Answers

1. **(1.5pt)** If $z \in \Omega_r$, then $d(z, \mathbb{C} \setminus \Omega) > r > 0$. Since any point z of $\mathbb{C} \setminus \Omega$ satisfies $d(z, \mathbb{C} \setminus \Omega) = 0$, we have $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus \Omega_r$ and thus $\Omega_r \subset \Omega$.

   For any $z \in \Omega_r$, the number $\epsilon = d(z, \mathbb{C} \setminus \Omega) - r$ is positive. If $|w - z| < \epsilon$ then for any $v \in \mathbb{C} \setminus \Omega$, $|z - v| \geq d(z, \mathbb{C} \setminus \Omega)$ and

   $$|w - v| \geq |z - v| - |w - z| > d(z, \mathbb{C} \setminus \Omega) - \epsilon = r.$$

   Consequently $D(z, \epsilon) \subset \Omega$ and Ω is open.

   Finally, if $z \in \Omega$, since Ω is open, $d = d(z, \mathbb{C} \setminus \Omega) > 0$, thus if $r = d/2$, the point z belongs to Ω_r. Consequently, $\Omega = \cup_{r>0} \Omega_r$.

2. **(1pt)** Let $\Omega = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < |\operatorname{Re} z| + 1\}$. This set is open: the function

   $$\phi : z \in \mathbb{C} \to |\operatorname{Re} z| + 1 - |\operatorname{Im} z| \in \mathbb{R}$$

   is continuous and Ω is the pre-image of the open set $\mathbb{R}_+^*$ by ϕ. Now, for any purely imaginary number iy of Ω, since $|iy - i| \leq 1$ or $|iy + i| \leq 1$, we have $d(iy, \mathbb{C} \setminus \Omega) \leq 1$. Therefore, no such point belongs to Ω_1. On the other hand, $d(-2, \mathbb{C} \setminus \Omega) = d(-2, \mathbb{C} \setminus \Omega) = 3\sqrt{2}/2 > 1$ and hence $-2 \in \Omega_1$ and $2 \in \Omega_1$.

   Assume that Ω_1 is connected. Since it is open, it is path-connected: there is a continuous function $\gamma : [0, 1] \to \Omega_1$ that joins -2 and 2. By the intermediate value theorem, there is a $t \in]0, 1[$ such that $\operatorname{Re} \gamma(t) = 0$. Since $\gamma(t) \in \Omega_1$, we have a contradiction. Consequently, Ω_1 is not connected.

3. **(2.5pt)** By definition of Ω_r, its complement satisfies

   $$\mathbb{C} \setminus \Omega_r = \{z \in \mathbb{C} \mid d(z, \mathbb{C} \setminus \Omega) \leq r\}.$$

   Thus, if $z \in \mathbb{C} \setminus \Omega_r$, since $\mathbb{C} \setminus \Omega$ is closed, there is a $w \in \mathbb{C} \setminus \Omega$ such that $|z - w| \leq r$. Since for any $\lambda \in [0, 1]$, the point $z_\lambda = \lambda w + (1 - \lambda) z$ satisfies $|z_\lambda - w| = (1 - \lambda)|z - w| \leq r$, we also have $z_\lambda \in \mathbb{C} \setminus \Omega_r$. Hence, $[w, z] \subset \mathbb{C} \setminus \Omega_r$.

   Assume that Ω is simply connected. Let $z \in \mathbb{C} \setminus \Omega_r$ and let γ be a closed path of Ω_r; since $\Omega_r \subset \Omega$, γ is also a closed path of Ω. Let $w \in \mathbb{C} \setminus \Omega$ such that $[w, z] \subset \mathbb{C} \setminus \Omega_r$. Since $[w, z]$ is connected and the function $\xi \in \mathbb{C} \setminus \Omega_r \mapsto \operatorname{ind}(\gamma, \xi)$ is locally constant, $\operatorname{ind}(\gamma, z) = \operatorname{ind}(\gamma, w)$. Since Ω is simply connected, $\operatorname{ind}(\gamma, w) = 0$. Hence, Ω_r is simply connected.

   Alternatively, assume that Ω_r is multiply connected. Let $\mathbb{C} \setminus \Omega_r = K \cup L$ where K is bounded and non-empty and $d(K, L) > 0$. Since $\Omega_r \subset \Omega$, $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus \Omega_r$. Let $K' = K \cap (\mathbb{C} \setminus \Omega)$ and $L' = L \cap (\mathbb{C} \setminus \Omega)$. Clearly, $\mathbb{C} \setminus \Omega = K' \cup L'$, K' is bounded and $d(K', L') > 0$. Now, let $z \in K$ and let $w \in \mathbb{C} \setminus \Omega$ such that $[w, z] \subset \mathbb{C} \setminus \Omega_r = K \cup L$. Every connected C

subset of $K \cup L$ that contains a point of K is included in K since for any $r \leq d(K, L)$, the dilation $C + B(0, r)$ doesn't intersect L. Since $[w, z]$ is a connected subset of $K \cup L$ and $z \in K$, we have $[w, z] \subset K$, thus K' is also non-empty. Finally, Ω is multiply connected.

The converse result is false: for any open subset Ω which is bounded and not simply connected, for r large enough, $\Omega_r = \varnothing$ which is simply connected.

4. **(2.5pt)** Assume that $\mathbb{C} \setminus \Omega$ is disconnected. Since Ω is bounded, for r large enough, the annulus $A(0, r + \infty)$, which is connected, is a subset of $\mathbb{C} \setminus \Omega$. Consider a dilation of $\mathbb{C} \setminus \Omega$ which is not (path-)connected; let V_∞ be the component of this dilation that contains the annulus above, while V is the (non-empty) union of the other components of the dilation. By construction, V and V_∞ are open and disjoints and V is bounded. The set

$$K = (\mathbb{C} \setminus \Omega) \cap V = (\mathbb{C} \setminus \Omega) \setminus V_\infty$$

is closed, non-empty and bounded; with

$$L = (\mathbb{C} \setminus \Omega) \cap V_\infty = (\mathbb{C} \setminus \Omega) \setminus V,$$

which is closed, we have $\mathbb{C} \setminus \Omega = K \cup L$. Thus $\mathbb{C} \setminus \Omega$ is multiply connected.

5. **(1.5pt)** Since Ω is bounded, there is a $r > 0$ such that $\Omega \subset K = \overline{D(0, r)}$. Let $a \in \mathbb{C}$ such that $|a| > r$; let $\epsilon > 0$ and $\hat{f} : \mathbb{C} \setminus \{a\}$ be a holomorphic function such that $|f - \hat{f}| \leq \epsilon/2$ on Ω. Since K is a compact subset of $D(0, |a|) \subset \mathbb{C} \setminus \{a\}$, the Taylor series expansion $\sum a_n (z - c)^n$ of $\hat{f}$ in $D(0, |a|)$ is uniformly convergent in K, thus there is a $m \in \mathbb{N}$ such that the polynomial

$$p(z) = \sum_{n=0}^{m} a_n (z - c)^n$$

satisfies $|\hat{f} - p| \leq \epsilon/2$ in K and hence in Ω. Consequently,

$$\forall\, z \in \Omega,\ |f(z) - p(z)| \leq |f(z) - \hat{f}(z)| + |\hat{f}(z) - p(z)| \leq \epsilon.$$

6. **(1.5pt)** Assume that there is a polynomial p such that $|f - p| < 1$ on Ω. Since Ω is bounded, the set $K = \overline{\Omega}$ is compact and since p is continuous on K, it is bounded on Ω. Given that

$$|f(z)| \leq |f(z) - p(z)| + |p(z)|$$

the function f is necessarily bounded on Ω.

Now all we have to do is to find a holomorphic function on Ω which is not bounded. Consider $f : z \mapsto 1/(z - a)$ where a is a point of the boundary of Ω (such a point exists since $\Omega \neq \varnothing$ and $\Omega \neq \mathbb{C}$); since Ω is open, $a \notin \Omega$ and the function f is defined and holomorphic on Ω. By construction there is a sequence $z_n \in \Omega$ such that $z_n \to a$ and thus

$$|f(z_n)| = \left|\frac{1}{z_n - a}\right| \to +\infty,$$

thus f is not bounded on Ω.

7. **(2pt)** If Ω is not simply connected, there is a closed rectifiable path γ of Ω and a point $a \in \mathbb{C} \setminus \Omega$ which is in the interior of γ, that is $\text{ind}(\gamma, a)$ is a non-zero integer. The function $z \mapsto 1/(z-a)$ is defined and holomorphic on Ω. Additionally

$$\frac{1}{2\pi} \int_\gamma f(z) dz = \text{ind}(\gamma, a) \in \mathbb{Z}^*.$$

Since the distance between $\gamma([0,1])$ and the complement of Ω is positive, there is a $r > 0$ such that $\gamma([0,1]) \subset \Omega_r$. Now if $\hat{f}$ is a polynomial such that $|f - \hat{f}| \leq \epsilon$ on Ω_r,

$$\left|\frac{1}{2\pi} \int_\gamma f(z)\, dz\right| \leq \left|\frac{1}{2\pi} \int_\gamma \hat{f}(z)\, dz\right| + \left|\frac{1}{2\pi} \int_\gamma (f - \hat{f})(z)\, dz\right|.$$

By Cauchy's theorem (the local version, in $\mathbb{C}$), the first integral in the right-hand side is zero. By the M-L inequality, the second one is dominated by $(\epsilon/2\pi) \times \ell(\gamma)$. Thus for $\epsilon < 2\pi/\ell(\gamma)$, we would have

$$\left|\frac{1}{2\pi} \int_\gamma f(z)\, dz\right| < 1.$$

Consequently $z \mapsto 1/(z-a)$ is holomorphic on Ω but cannot be locally uniformly approximated by polynomials.

8. **(3pt)** The condition $|w - z| < r/2$ yields

$$\{z\} \subset \overline{D(w, r/2)} = \mathbb{C} \setminus A(w, r/2, +\infty).$$

or equivalently, $A = A(w, r/2, +\infty) \subset \mathbb{C} \setminus \{z\}$. Additionally, any $v \in \overline{\Omega_r}$ satisfies $d(v, \mathbb{C} \setminus \Omega) \geq r$ and in particular $|v - z| \geq r$. Since $|w - z| < r/2$,

$$|v - w| \geq |v - z| - |w - z| > r - r/2 = r/2,$$

hence $v \in A(w, r/2, +\infty)$. Consequently, $\overline{\Omega_r} \subset A$.

Since $\chi_r(z) = 1$, for any $\epsilon > 0$, there is a function $\hat{f}_z$ holomorphic in $\mathbb{C} \setminus \{z\}$ such such that $|f - \hat{f}| \leq \epsilon/2$ on Ω_r. Now, since the annulus A is included in the domain of definition of $\hat{f}_z$, we have the Laurent series expansion

$$\forall\, v \in A,\ \hat{f}_z(v) = \sum_{n=-\infty}^{+\infty} a_n (v - w)^n.$$

This expansion is locally uniformly convergent; since $\overline{\Omega_r}$ is compact and included in A, there is a natural number m such that the function $\hat{f}_w(v) := \sum_{n=-m}^{m} a_n(v-w)^n$ – which is holomorphic on $\mathbb{C}\setminus\{w\}$ – satisfies $|\hat{f}_z - \hat{f}_w| \leq \epsilon/2$ on $\overline{\Omega}_r$. Finally, on Ω_r, we have

$$|f - \hat{f}_w| \leq |f - \hat{f}_z| + |\hat{f}_z - \hat{f}_w| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

and thus $\chi(w) = 1$.

9. **(1pt)** We know that if $\chi(z) = 1$ and $|w - z| < r/2$, then $\chi(w) = 1$. Now, by contraposition of this property, if $\chi(w) = 0$ and $|w - z| < r/2$, we have $\chi(z) = 0$. Therefore χ is locally constant. Now since Ω is simply connected and bounded, by question 4, $\mathbb{C} \setminus \Omega$ is connected. Since $\chi : \mathbb{C} \setminus \Omega \to \{0, 1\}$ is locally constant, if $\chi(a) = 1$ for some $a \in \mathbb{C} \setminus \Omega$, $\chi = 1$ on $\mathbb{C} \setminus \Omega$.

10. **(1pt)** If $f : \Omega \to \mathbb{C}$ has locally uniform approximations among the holomorphic functions defined on $\mathbb{C} \setminus \{a\}$, then for any $r > 0$ $\chi_r(a) = 1$. By the previous question, $\chi_r(b) = 1$ for any $b \in \mathbb{C} \setminus \Omega$ and thus f has locally uniform approximations among the holomorphic functions defined on $\mathbb{C} \setminus \{b\}$. We can select a b that is arbitrarily large, thus by question 5, f has locally uniform approximations among polynomials.

11. **(4pt)** We proceed by induction. The result is plain if $n = 1$; now assume that the result holds for a given $n \geq 1$ and let $\hat{f} : \mathbb{C}\setminus\{a_1, \ldots, a_n, a_{n+1}\} \to \mathbb{C}$ be holomorphic. Since a_{n+1} is an isolated singularity of f, there is a $r > 0$ such that $A(a_{n+1}, 0, r)$ is a non-empty annulus included in the domain of definition of f. Let $\sum_{k=-\infty}^{+\infty} b_k(z - a_{n+1})^k$ be the Laurent series expansion of $\hat{f}$ in this annulus and define

$$\hat{f}_{n+1}(z) = \sum_{n=-\infty}^{-1} b_k(z - a_{n+1})^k$$

Since there are only negative powers, the sum is convergent (and holomorphic) in $\mathbb{C} \setminus \{a_{n+1}\}$. Now since in the annulus

$$\hat{f}(z) - \hat{f}_{n+1}(z) = \sum_{n=0}^{+\infty} b_k(z - a_{n+1})^k$$

the point a_{n+1} is a removable singularity of the function $\hat{f} - \hat{f}_k$ that may thus be extended holomorphically to $\mathbb{C} \setminus \{a_1, \ldots, a_n\}$. We may apply the induction hypothesis to this extension; we get holomorphic functions $\hat{f}_k : \mathbb{C} \setminus \{a_k\} \to \mathbb{C}$ for $k = 1, \ldots, n$ such that

$$\forall\, z \in \mathbb{C} \setminus \{a_1, \ldots, a_{n+1}\},\ \hat{f}(z) - \hat{f}_{n+1}(z) = \hat{f}_1(z) + \cdots + \hat{f}_n(z)$$

which is the induction hypothesis at stage $n + 1$.

Now the corollary: assume that for any $r > 0$ and any $\epsilon > 0$, there is a holomorphic function $\hat{f} : \mathbb{C} \setminus \{a_1, \dots, a_n\} \to \mathbb{C}$ such that $|f - \hat{f}| \leq \epsilon/2$ in Ω_r. Let $\hat{f}_k : \mathbb{C} \setminus \{a_k\} \to \mathbb{C}$ for $k = 1, \dots, n$ be such that

$$\forall\, z \in \mathbb{C} \setminus \{a_1, \dots, a_n\},\ \hat{f}(z) = \hat{f}_1(z) + \cdots + \hat{f}_n(z).$$

Since the restriction of every $\hat{f}_k$ to Ω is locally uniformly approximated by holomorphic functions on $\mathbb{C} \setminus \{a_k\}$, by question 9, it is locally uniformly approximated by polynomials, so there is a polynomial p_k such that $|\hat{f}_k - p_k| \leq \epsilon/2n$ on Ω_r. Let $p = \sum_{k=1}^{n} p_k$; on Ω_r, we have

$$|f - p| \leq |f - \hat{f}| + \sum_{k=1}^{n} |\hat{f}_k - p_k| \leq \epsilon.$$

Thus, f is locally uniformly approximated by polynomials.

Problem L – Answers

1. **(1pt)** Since

$$\mathcal{L}_\mu[f](s) = \int_{\mathbb{R}_+} f(a + t\lambda u) e^{-s(a+t\lambda u)} (\lambda u)\, dt,$$

if $\mathcal{L}_\mu[f](s)$ is defined, the change of variable $\tau = \lambda t$ yields

$$\mathcal{L}_\mu[f](s) = \int_{\mathbb{R}_+} f(a + \tau u) e^{-s(a+\tau u)} u\, d\tau,$$

thus $\mathcal{L}_\lambda[f](s)$ is defined and $\mathcal{L}_\mu[f](s) = \mathcal{L}_\lambda[f](s)$. Conversely, if $\mathcal{L}_\lambda[f](s)$ is defined, the same argument with $1/\lambda$ instead of λ shows that $\mathcal{L}_\mu[f](s)$ is also defined (and obviously $\mathcal{L}_\mu[f](s) = \mathcal{L}_\lambda[f](s)$).

2. **(2pt)** The set $\Pi(u, \sigma)$ is an open half-plane: if $u = |u|e^{i\theta}$ and $s = x + iy$, then the condition $\mathrm{Re}(su) > \sigma|u|$ is equivalent to

$$x \cos\theta - y \sin\theta > \sigma.$$

The Laplace transform of f along γ satisfies

$$\begin{aligned}\mathcal{L}_\gamma[f](s) &= \int_{\mathbb{R}_+} f(a + tu) e^{-s(a+tu)} u\, dt \\ &= (e^{-sa} u) \int_{\mathbb{R}_+} f(a + tu) e^{-(su)t}\, dt\end{aligned}$$

and thus

$$\mathcal{L}_\gamma[f](s) = (e^{-sa} u) \mathcal{L}[t \mapsto f(a + tu)](su).$$

The function $t \in \mathbb{R}_+ \mapsto f(a+tu)$ is locally integrable (it is continuous since f is holomorphic) and since for any $t \geq 0$

$$-|a| + t|u| \leq |a+tu| \leq |a| + t|u|,$$

we have

$$|f(a+tu)| \leq \kappa e^{\sigma|a+tu|} \leq \kappa \max(e^{-\sigma|a|}, e^{\sigma|a|}) e^{(\sigma|u|)t}$$

therefore $t \geq 0 \mapsto |f(a+tu)|e^{-\sigma^+ t}$ is integrable whenever $\sigma^+ > \sigma|u|$. Hence the Laplace transform $\mathcal{L}[t \mapsto f(a+tu)]$ is defined and holomorphic on the set $\Pi(u,\sigma) = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma|u|\}$ and $\mathcal{L}_\gamma[f]$ as well, as a composition and product of holomorphic functions.

3. **(3pt)** For a $s \in \mathbb{C}$, the set of u such that $s \in \Pi(u,\sigma)$ is open as the preimage of the open set $]0, +\infty[$ by the continuous function $u \in U \mapsto \mathrm{Re}(su) - \sigma|u|$.

   Let $\psi : U_s \times \mathbb{R}_+ \to \mathbb{C}$ be defined as

   $$\psi(u,t) = f(a+tu)e^{-(a+tu)s}u.$$

   Since

   $$|f(a+tu)| \leq \kappa e^{\sigma|a+tu|} \leq \kappa_1 e^{t|u|\sigma}$$

   with $\kappa_1 = \kappa \max(e^{-\sigma|a|}, e^{\sigma|a|})$ and

   $$|e^{-(a+tu)s}| = e^{-\mathrm{Re}((a+tu)s)} = e^{-\mathrm{Re}(as)}e^{-t\mathrm{Re}(su)},$$

   we end up with

   $$|\psi(u,t)| \leq (\kappa_1 e^{-\mathrm{Re}(as)}|u|)e^{t(\sigma|u| - \mathrm{Re}(su))}.$$

   Since for any $u \in U_s$

   $$\mathcal{L}_\gamma[f](s) = \int_{\mathbb{R}_+} \psi(u,t)\, dt$$

   we check the assumptions of the complex-differentiation under the integral sign theorem:

   - For every $u \in U_s$, the function $t \in \mathbb{R}_+ \mapsto \psi(u,t)$ is Lebesgue measurable (it is actually continuous since f is holomorphic).
   - If $s \in \Pi(u_0,\sigma)$, since $\epsilon := \mathrm{Re}(su_0) - \sigma|u_0| > 0$, by continuity there is a $0 < r < |u_0|$ such that $|u - u_0| \leq r$ ensures $\mathrm{Re}(su) - \sigma|u| \geq \epsilon/2$. Let κ_2 be an upper bound of $\kappa_1 e^{-\mathrm{Re}(as)}|u|$ when $|u - u_0| \leq r$. The bound on ψ that we have derived above yields

     $$|\psi(u,t)| \leq \kappa_2 e^{-t\epsilon/2}$$

     and the right-hand side of this inequality is a Lebesgue integrable function of t.

- For every $t \in \mathbb{R}_+$, it is plain that the function $u \in U_s \mapsto \psi(u,t)$ is holomorphic.

Consequently $\mathcal{L}_\gamma[f](s)$ is a complex-differentiable function of u.

4. **(1.5pt)** First method: since we know that the complex-derivative of $\mathcal{L}_\gamma[f](s)$ exists with respect to u, we can apply the chain rule to the differentiable function

$$\chi : \lambda \in \mathbb{R}_+^* \to \mathcal{L}_\mu[f](s) \text{ with } \mu(t) = a + t(\lambda u)$$

at $t = 1$: it yields

$$\frac{d\chi}{d\lambda}(1) = \frac{d}{du}\mathcal{L}_\mu[f](s) \times \frac{d(\lambda u)}{d\lambda}(1) = \frac{d}{du}\mathcal{L}_\mu[f](s) \times u.$$

Since by question 1 we know that the function χ is constant, we conclude that $\frac{d}{du}\mathcal{L}_\mu[f](s) = 0$.

Second method: we use the result of the differentiation under the integral sign. It provides

$$\frac{d}{du}\mathcal{L}_\mu[f](s) = \int_{\mathbb{R}_+} \frac{\partial \psi}{\partial u}(u,t)\,dt.$$

Since $\psi(u,t) = g(tu)u$ with $g(z) = f(a+z)e^{-s(a+z)}$, we have

$$\begin{aligned}\frac{\partial \psi}{\partial u}(u,t) &= \frac{d}{du}(g(tu)u) = g'(tu)tu + g(tu) = \frac{dg(tu)}{dt}t + g(tu)\\ &= \frac{d}{dt}(g(tu)t)\end{aligned}$$

and thus

$$\begin{aligned}\int_0^r \frac{\partial \psi}{\partial u}(u,t)\,dt &= \int_0^r \frac{d}{dt}(g(tu)t)\,dt = [g(tu)t]_0^r\\ &= f(a+ru)e^{-s(a+ru)}r\end{aligned}$$

Since $\epsilon := \mathrm{Re}(su) - \sigma|u| > 0$

$$|f(a+ru)e^{-s(a+ru)}r| \leq |\psi(u,r)r/u| \leq (\kappa_1 e^{-\mathrm{Re}(as)})e^{-\epsilon r}r.$$

The right-hand side of this inequality converges to 0 when $r \to +\infty$. Therefore, by the dominated convergence theorem

$$\frac{d}{du}\mathcal{L}_\mu[f](s) = \int_{\mathbb{R}_+} \frac{\partial \psi}{\partial u}(u,t)\,dt = \lim_{r\to+\infty}\int_0^r \frac{\partial \psi}{\partial u}(u,t)\,dt = 0.$$

5. **(2pt)** Since the Laplace transform of $t \in \mathbb{R}_+ \to 1/(t+1)$ is given by

$$F(s) = \int_0^{+\infty} \frac{e^{-st}}{t+1}\,dt,$$

the change of variable $t+1 \to t$ yields

$$F(s) = \int_1^{+\infty} \frac{e^{-s(t-1)}}{t}\, dt = e^s \int_1^{+\infty} \frac{e^{-st}}{t}\, dt.$$

If s is equal to the real number $x > 0$, the change of variable $xt \to t$ then provides

$$F(x) = e^x \int_1^{+\infty} \frac{e^{-xt}}{xt}\, d(xt) = e^x \int_x^{+\infty} \frac{e^{-t}}{t}\, dt$$

and thus $E_1(x) = e^{-x}F(x)$. Since $t \in \mathbb{R}_+ \to 1/(t+1)e^{-\sigma^+ t}$ is integrable whenever $\sigma^+ > 0$ the Laplace transform F of $t \mapsto 1/(t+1)$ is defined and holomorphic on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0\}$. Thus, the function $G : s \mapsto e^{-s}F(s)$ extends holomorphically E_1 on this open right-hand plane. Any other function $\tilde{G}$ with the same property would be equal to G on $\mathbb{R}_+^*$ and thus every positive real number x would be a non-isolated zero of $G - \tilde{G}$. By the isolated zeros theorem, since the open right-hand plane is connected, G and $\tilde{G}$ are necessarily equal.

6. **(2pt)** For any $u \in U$, since $\operatorname{Re} u > 0$, for any $t \geq 0$, $\operatorname{Re}(\gamma(t)) = t \operatorname{Re} u \geq 0$. Since for any z such that $\operatorname{Re} z \geq 0$, $|z+1| \geq 1$, we have $|f(z)| = 1/|z+1| \leq 1$ and thus

$$\forall\, z \in \gamma(\mathbb{R}_+),\ |f(z)| = \kappa e^{\sigma |z|}$$

with $\kappa = 1$ and $\sigma = 0$.

By definition, for any $s \in \mathbb{C} \setminus \mathbb{R}_-$, U_s is the set of all directions u such that $\operatorname{Re}(u) > 0$ and $\operatorname{Re}(su) > 0$. To be more explicit, if α denotes the argument of u in $[-\pi, \pi[$ and β the argument of s in $]-\pi, \pi[$, this is equivalent to

$$-\pi/2 < \alpha < \pi/2 \ \text{ and } \ -\pi/2 < \alpha + \beta < \pi/2$$

The points $u = re^{i\alpha}$ that satisfy these constraints form an open sector: $r > 0$ is arbitrary and if $\beta \geq 0$, α is subject to

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2} - \beta$$

and if $\beta < 0$

$$-\frac{\pi}{2} - \beta < \alpha < \frac{\pi}{2}.$$

In any case, since $|\beta| < \pi$, the set of admissible arguments α is not empty.

7. **(3pt)** It is plain that

$$\operatorname{Re}(u_\theta) = \operatorname{Re}((1-\theta)u_0 + \theta u_1) = (1-\theta)\operatorname{Re}(u_0) + \theta \operatorname{Re}(u_1).$$

and that for any $s \in \mathbb{C}$

$$\operatorname{Re}(su_\theta) = \operatorname{Re}(s((1-\theta)u_0 + \theta u_1)) = (1-\theta)\operatorname{Re}(su_0) + \theta \operatorname{Re}(su_1).$$

Since $u \in U_s$ if and only if $\mathrm{Re}(u) > 0$ and $\mathrm{Re}(su) > 0$, if $u_0 \in U_s$ and $u_1 \in U_s$ then $u_\theta \in U_s$ for any $\theta \in [0,1]$.

The function $u \in U_s \mapsto \mathcal{L}_\gamma[f](s)$ is holomorphic, hence the composition of

$$\theta \in [0,1] \mapsto u_\theta \in U_s \ \text{ and } \ u \in U_s \mapsto \mathcal{L}_\gamma[f](s)$$

is continuously differentiable,

$$\mathcal{L}_{\gamma_0}[f](s) - \mathcal{L}_{\gamma_1}[f](s) = \int_0^1 \frac{d}{d\theta}\mathcal{L}_{\gamma_\theta}[f](s)\, d\theta$$

and

$$\frac{d}{d\theta}\mathcal{L}_{\gamma_\theta}[f](s) = \frac{d\mathcal{L}_\gamma[f](s)}{du}|_{u=u_\theta} \times \frac{du_\theta}{d\theta}$$

which is zero by question 4. Hence $\mathcal{L}_{\gamma_0}[f](s) = \mathcal{L}_{\gamma_1}[f](s)$; the definition of G is unambiguous.

8. **(4pt)** For $k \in \{0,1\}$ and any $r \geq 0$,

$$\int_{\gamma_k^r} f(z)e^{-sz}\, dz = \int_0^1 f((tr)u_k)e^{-s((tr)u_k)} r u_k \, dt = \int_0^r f(tu_k)e^{-s(tu_k)} u_k \, dt$$

thus by the dominated convergence theorem

$$\mathcal{L}_{\gamma_1}[f](s) - \mathcal{L}_{\gamma_0}[f](s) = \lim_{r \to +\infty} \left(\int_{\gamma_1^r} f(z)e^{-sz} dz - \int_{\gamma_0^r} f(z)e^{-sz} dz \right).$$

Since $(\gamma_0^r)^{\leftarrow}$ and γ_1^r are consecutive, with the path $\mu_r = (\gamma_0^r)^{\leftarrow} \,|\, \gamma_1^r$, this is equivalent to

$$\mathcal{L}_{\gamma_1}[f](s) - \mathcal{L}_{\gamma_0}[f](s) = \lim_{r \to +\infty} \int_{\mu_r} f(z)e^{-sz} dz.$$

Now, the path ν_r defined by $\nu_r(\theta) = (1-\theta)ru_0 + \theta r u_1$ is such that $\mu_r \,|\, \nu_r^{\leftarrow}$ is a closed rectifiable path in U_s which is simply connected. Therefore by Cauchy's integral theorem

$$\int_{\mu_r} f(z)e^{-sz} dz = \int_{\nu_r} f(z)e^{-sz} dz.$$

For any $\theta \in [0,1]$, we have $\mathrm{Re}(su_\theta) = (1-\theta)\mathrm{Re}(su_0) + \theta \mathrm{Re}(su_1)$, thus

$$\epsilon := \min_{\theta \in [0,1]} \mathrm{Re}(su_\theta) > 0$$

and

$$|f(\nu_r(\theta))e^{-s\nu_r(\theta)}| \leq \kappa e^{-\mathrm{Re}(sru_\theta)} \leq \kappa e^{-\epsilon r}.$$

Given that $\ell(\nu_r) = r|u_1 - u_0|$, the M-L inequality provides

$$\left| \int_{\nu_r} f(z) e^{-sz} dz \right| \leq \kappa e^{-\epsilon r} r |u_1 - u_0|$$

and thus

$$\lim_{r \to +\infty} \int_{\nu_r} f(z) e^{-sz} dz = 0.$$

Finally, $\mathcal{L}_{\gamma_1}[f](s) = \mathcal{L}_{\gamma_0}[f](s)$: the definition of G is unambiguous.

9. **(1pt)** By construction the function

$$s \in \mathbb{C} \setminus \mathbb{R}_- \mapsto e^{-s} G(s)$$

is holomorphic (since every $\mathcal{L}_\gamma$ is holomorphic, G is holomorphic); It extends $s \mapsto e^{-s} F(s)$, thus it also extends E_1. Since $\mathbb{C} \setminus \mathbb{R}_-$ is connected, by the isolated zeros theorem, this extension is unique (the argument is identical to the one used in question 6).