Analytic Functions

Sébastien Boisgérault, Mines ParisTech, under CC BY-NC-SA 4.0

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1. Show that the function

$$f : x \in \mathbb{R} \mapsto \frac{1}{1+x^2}$$

is analytic.

2. Determine for any $x_0 \in \mathbb{R}$ the open interval of convergence of its Taylor series expansion at x_0.

Answers

1. The function f is the restriction of the holomorphic function

$$f^* : z \in \mathbb{C} \setminus \{i, -i\} \mapsto \frac{1}{1+z^2}.$$

2. For any $x_0 \in \mathbb{R}$, the disk

$$D(x_0) = D(x_0, \sqrt{1+x_0^2})$$

is included in $\mathbb{C} \setminus \{i, -i\}$. The radius of the disk of convergence of the Taylor expansion of f^* at x_0 is therefore at least $\sqrt{1+x_0^2}$; it cannot exceed this threshold: otherwise, the sum $g(z)$ of its Taylor series would be defined and holomorphic in an open set that contains $\overline{D}_0$ and therefore bounded on D_0; but g and f^* are identical on D_0 where f^* is unbounded. Finally, as the Taylor expansion of f at x_0 has the same coefficient as the Taylor expansion of f^* at x_0, the open interval of convergence of f^* at x_0 is

$$\left]x_0 - \sqrt{1+x_0^2}, x_0 + \sqrt{1+x_0^2}\right[.$$

Analytic Functions of a Real Variable

Questions

1. Show that the function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(x) = \left| \begin{array}{rl} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{array} \right.$$

is smooth but is not analytic.

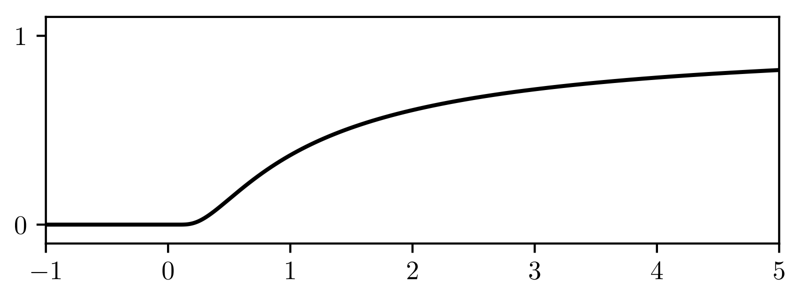

Figure 1: The graph of a function which is smooth, but not analytic.

2. Let K be an compact interval of $\mathbb{R}$ and $f : K \to \mathbb{C}$ be a smooth function. Show that f is analytic if and only if there are positive constants $\alpha > 0$ and $r > 0$ such that

$$\forall\, x \in K, \forall\, n \in \mathbb{N},\ |f^{(n)}(x)| \leq \alpha\, r^n n!$$

Answers

1. By induction, for any $x > 0$, $f^{(n)}(x) = g_n(x)e^{-1/x}$ where g_n is a rational function, defined for $x > 0$ by

$$g_0(x) = 1\ \wedge\ \forall\, n \in \mathbb{N},\ g_{n+1}(x) = g_n'(x) + \frac{g_n(x)}{x^2}.$$

On the other hand, for $x \leq 0$, the n-th order derivative (left-derivative at $x = 0$) of f at x is defined and equal to 0. To prove that f is smooth, we now have to prove that the right, n-th order derivative of f at 0 exists and is equal to its left derivative, that is zero. We may proceed by induction: suppose that $f^{(n)}(0)$ exists and is zero; then

$$\frac{f^{(n)}(h) - f^{(n)}(0)}{h} = \frac{1}{h}\int_0^h f^{(n+1)}(x)dx.$$

But for any n, we have

$$\lim_{x \to 0^+} f^{(n+1)}(x) = \lim_{x \to 0^+} g_{n+1}(x)e^{-1/x} = 0,$$

hence the right-hand side of the equation tends to zero when $h \to 0^+$: the $n+1$-th order right derivative of f exists at $x = 0$ and is equal to zero.

Now, the function f cannot be analytic: given that its derivatives at $x = 0$ are zero at all order, its Taylor series expansion at the origin is zero. The function f would be zero in a neighbourhood of the origin, and this property does not hold.

2. If the function f is smooth, for any real numbers c and y in K, the Taylor formula with integral remainder is applicable at any order n:

$$f(y) = \sum_{p=0}^{n} \frac{f^{(p)}(c)}{p!}(y-c)^p + \int_c^y \frac{f^{(n+1)}(x)}{n!}(x-c)^n dx.$$

If there exist $\alpha > 0$ and $r > 0$ such that the inequality

$$\forall\, x \in K,\ |f^{(n)}(x)| \leq \alpha\, r^n n!$$

holds, the remainder satisfies

$$\begin{aligned}\left|\int_c^y \frac{f^{(n+1)}(x)}{n!}(x-c)^n dx\right| &\leq \alpha\, r^{n+1}\frac{(n+1)!}{n!}\left|\int_c^y (x-c)^n\, dx\right|. \\ &= \alpha\,(r|y-c|)^{n+1}\end{aligned}$$

Thus, if $|y - c| < 1/r$, the Taylor expansion of f at y centered on c is convergent. As c is an arbitrary point of I, the function f is analytic.

Conversely, if f is analytic, it has a holomorphic extension – that we may still denote f – to some open neighbourhood U of K. The distance d between K and $\mathbb{C} \setminus U$ is positive: for any $c \in K$, the disk $D(c, d)$ is included in U. Let r be a positive radius smaller than d and α be an upper bound of $|f|$ on $K + \overline{D}(0, r)$; for any natural number n, we have

$$\left| \frac{f^{(n)}(c)}{n!} \right| = \left| \frac{1}{i2\pi} \int_{r[\circlearrowleft]+c} \frac{f(z)}{(z-c)^{n+1}} dz \right| \leq \alpha \left(r^{-1} \right)^n$$

which concludes the proof.

Periodic Analytic Functions

Notations. In this exercise, $\mathbb{U}$ is the unit circle centered at the origin:

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

For any radius $0 \leq r < 1$, we define the annulus

$$A_r = A(0, r, 1/r) = \{z \in \mathbb{C} \mid r < |z| < 1/r\}$$

(with the convention that $A_0 = \mathbb{C}^*$). For any $0 < \epsilon \leq +\infty$, the notation Ω_ϵ refers to the horizontal strip

$$\Omega_\epsilon = \{z \in \mathbb{C} \mid |\mathrm{Im}\ z| < \epsilon\}.$$

Questions

Let $f : \mathbb{U} \to \mathbb{C}$ be a function with an analytic extension in some open neighbourhood U of $\mathbb{U}$.

1. Prove that there is an annulus A_r such that $A_r \subset U$.
2. Let $g : t \in \mathbb{R} \mapsto f(e^{it})$; show that g is a 2π-periodic analytic function.

Conversely, let $g : t \in \mathbb{R} \to \mathbb{C}$ be a 2π-periodic analytic function.

3. Show there is an analytic extension g^* of g on some strip Ω_ϵ, such that

$$\forall\, z \in \Omega_\epsilon,\ g^*(z + 2\pi) = g^*(z).$$

4. Show that there exist a function $f : \mathbb{U} \to \mathbb{C}$ with an analytic extension in some open neighbourhood of $\mathbb{U}$ such that

$$\forall\, t \in \mathbb{R},\ g(t) = f(e^{it}).$$

Answers

1. The set $\mathbb{U}$ is compact and the set $\mathbb{C} \setminus U$ is closed; their intersection is empty, thus the distance $d = d(\mathbb{U}, \mathbb{C} \setminus U)$ is positive (it may be $+\infty$ if $U = \mathbb{C}$). On the other hand, for any $r < 1$,

$$d(\mathbb{U}, \mathbb{C} \setminus A_r) = \min(1 - r, 1/r - 1) = 1 - r.$$

   Thus, for any r such that $1 - r < d$, the annulus A_r is a subset of U.

2. The 2π-periodicity of g is clear: for any $t \in \mathbb{R}$,

$$g(t + 2\pi) = f(e^{i(t+2\pi)}) = f(e^{it}e^{i2\pi}) = f(e^{it}) = g(t).$$

   The assumption on f and the result of the previous question provide a holomorphic extension $f^* : A_r \to \mathbb{C}$ to $f : \mathbb{U} \to \mathbb{C}$ for some $r < 1$. Now,

$$|e^{iz}| = e^{\mathrm{Re}\ iz} = e^{-\mathrm{Im}\ z},$$

   thus if $|\mathrm{Im}\ z| < \ln 1/r$, then $\ln r < -\mathrm{Im}\ z < \ln 1/r$ which yields $r < |e^{iz}| < 1/r$. Therefore, if we set $\epsilon = \ln 1/r > 0$, we have

$$\forall\, z \in \mathbb{C},\ (z \in \Omega_\epsilon \ \Rightarrow\ e^{iz} \in A_r).$$

   Consequently, setting $g^*(z) = f^*(e^{iz})$ defines a function g^* on Ω_ϵ; it is an extension of $g : \mathbb{R} \to \mathbb{C}$ and it is holomorphic as the composition of holomorphic functions. Therefore, the function $g : \mathbb{R} \to \mathbb{C}$ is analytic.

   Alternate proof. Consider the Laurent series expansion of f^* in A_r:

$$f^*(z) = \sum_{n=-\infty}^{+\infty} a_n z^n.$$

   For any real numbers t_0 and t, we have

$$(e^{it})^n = e^{int} = e^{int_0}e^{in(t-t_0)} = e^{int_0} \sum_{m=0}^{+\infty} \frac{1}{m!} i^m n^m (t - t_0)^m,$$

   therefore

$$g(t) = f(e^{it}) = \sum_{n=-\infty}^{+\infty} a_n e^{int_0} \left[\sum_{m=0}^{+\infty} \frac{1}{m!} i^m n^m (t - t_0)^m \right].$$

   We can change the order of the summation in this double series if

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{Z}} \left| a_n e^{int_0} \frac{1}{m!} i^m n^m (t - t_0)^m \right| < +\infty$$

The general term of this double series satisfies

$$\left| a_n e^{int_0} \frac{1}{m!} i^m n^m (t-t_0)^m \right| \leq |a_n| \frac{1}{m!} |n|^m |t-t_0|^m$$

hence the sum is bounded by

$$\sum_{n=-\infty}^{+\infty} |a_n| (e^{|t-t_0|})^{|n|} \leq \sum_{n=-\infty}^{+\infty} |a_n| (e^{t-t_0})^n + \sum_{n=-\infty}^{+\infty} |a_n| (e^{-(t-t_0)})^n.$$

The Laurent series expansion of f^* is absolutely convergent in A_r, hence the sums in the right-hand side of this inequality are finite if

$$r < e^{t-t_0} < 1/r \text{ and } r < e^{-(t-t_0)} < 1/r$$

that is if $|t-t_0| < \epsilon = \ln 1/r$. After the change in the order of the summation, we end up with:

$$\forall\, t \in \mathbb{R},\ |t-t_0| < \epsilon \ \Rightarrow\ g(t) = \sum_{m=0}^{+\infty} b_m (t-t_0)^m$$

where

$$b_m = \left[\sum_{n=-\infty}^{+\infty} a_n e^{int_0} \frac{1}{m!} i^m n^m \right],$$

hence the function g is analytic.

3. The function g is analytic; let g^0 be an analytic extension of g in some open neighbourhood Ω of $\mathbb{R}$. However, if the distance between $\mathbb{R}$ and $\mathbb{C} \setminus \Omega$ is equal to zero – it may happen as both sets but neither of them is compact – then Ω contains no strip Ω_ϵ.

   Let's build a new analytic extension g^* on such a strip from g^0. First, the set Ω contains some open tubular neighbourhood V_ϵ of $[0, 2\pi]$ for any $\epsilon > 0$ small enough:

   $$V_\epsilon = \{z \in \mathbb{C} \mid d(z, [0, 2\pi]) < \epsilon\} \subset \Omega.$$

   Indeed, $[0, 2\pi]$ is compact, $\mathbb{C} \setminus \Omega$ is closed and their intersection is empty, hence $d(\mathbb{C} \setminus \Omega, [0, 2\pi]) > 0$; any ϵ smaller than (or equal to) this distance is admissible.

   Consider the function g^* defined on Ω_ϵ by

   $$g^*(z) = g^0(z + 2\pi k) \text{ if } k \in \mathbb{Z} \text{ and } z + 2\pi k \in V_\epsilon.$$

   It is plain that g^* is analytic and extends g to Ω_ϵ; by construction it also satisfies the property

   $$\forall\, z \in \Omega_\epsilon,\ g^*(z + 2\pi) = g^*(z).$$

The only point to check is that this definition is unambiguous, as we may have for some z several integers k and ℓ such that $z_k = z + 2\pi k \in V_\epsilon$ and $z_\ell = z + 2\pi\ell \in V_\epsilon$. Assume for example that $k < \ell$; in this case $z_k \in D(0,\epsilon)$ and $\ell = k+1$, *i.e.* $z_\ell = z_k + 2\pi$. The functions

$$w \in D(0,\epsilon) \mapsto g^0(w) \;\text{ and }\; w \in D(0,\epsilon) \mapsto g^0(w+2\pi)$$

are holomorphic and identical on $]-\epsilon, \epsilon[$; by the isolated zeros theorem, they are identical on $D(0,\epsilon)$ (which is connected) and in particular $g(z_k) = g(z_\ell)$. The definition of g^* is actually unambiguous.

4. To answer the question, we exhibit an analytic function $f^* : A_r \to \mathbb{C}$ with $\epsilon = \ln 1/r$ (or equivalently $r = e^{-\epsilon}$) such that

$$\forall\, z \in A_r,\; f^*(e^{iz}) = g^*(z).$$

For any $w \in \mathbb{C}^*$, there is a solution z_0 to the equation

$$e^{iz} = w,\; z \in \mathbb{C}$$

and the other solutionss are $z_0 + 2\pi k$, for $k \in \mathbb{Z}$. Additionally, if $w \in A_r$, then $z \in \Omega_\epsilon$ with $\epsilon = \ln 1/r$. We may define $f^* : A_r \to \mathbb{C}$ by

$$f^*(w) = g^*(z),\; e^{iz} = w.$$

This definition is unambiguous: two z that correspond to the same w differ from a multiple of 2π, but g^* is 2π-periodic hence the right-hand sides of this definition are equal.

Let's prove that f^* is analytic. Let w_0 in A_r and z_0 such that $e^{iz_0} = w_0$, the expression

$$\phi(w) = -i \log \frac{w}{w_0} + z_0$$

defines an analytic function ϕ in an neighbourhood of w_0. It satisfies $e^{i(\phi(w)-z_0)} = w/w_0$, thus

$$e^{i\phi(w)} = w.$$

Consequently, in a neighbourhood of w_0,

$$f^*(w) = g^*(\phi(w))$$

and f^* is holomorphic – locally everywhere – as a composition of holomorphic functions.