Linear Prediction

Digital Audio Coding

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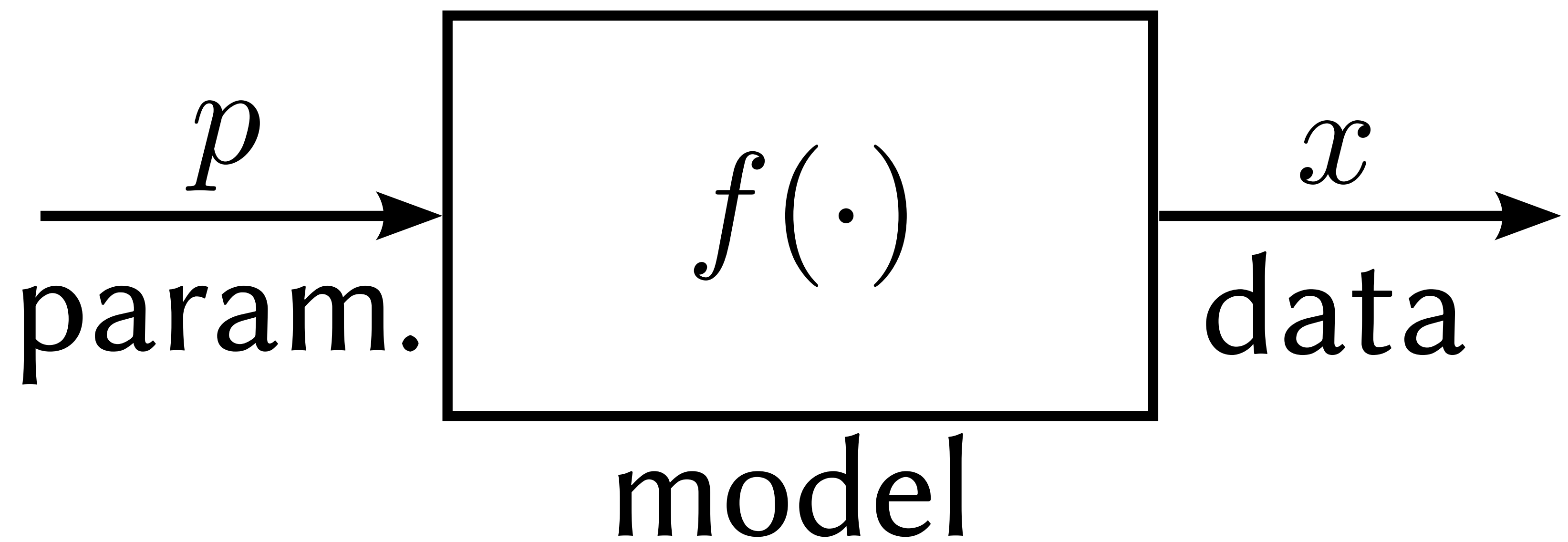
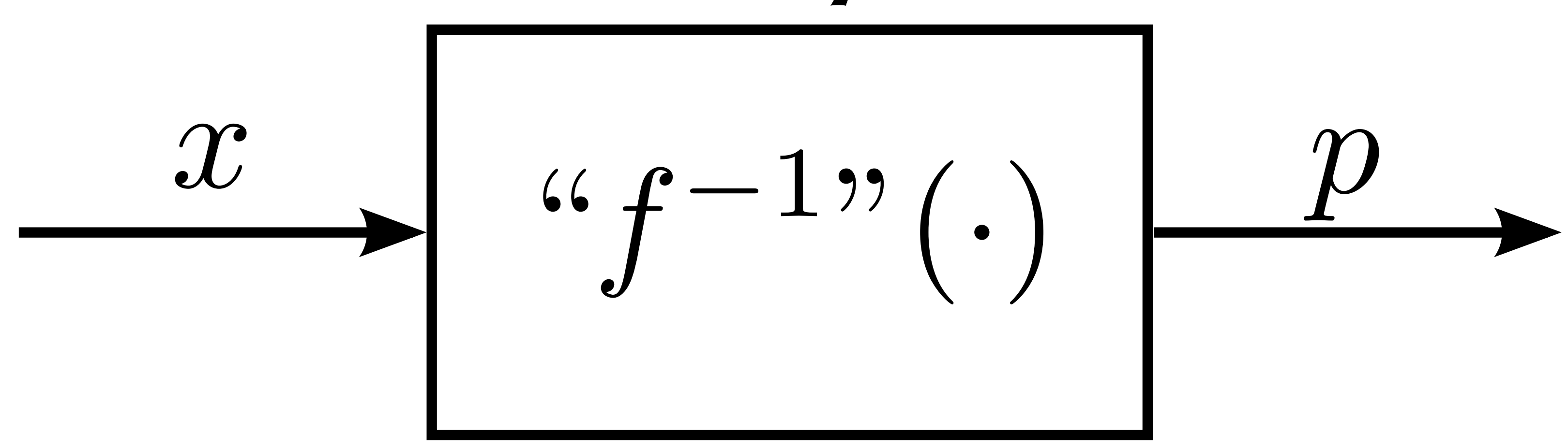
Parametric Models

Synthesis



Assume that f is onto ; invert with

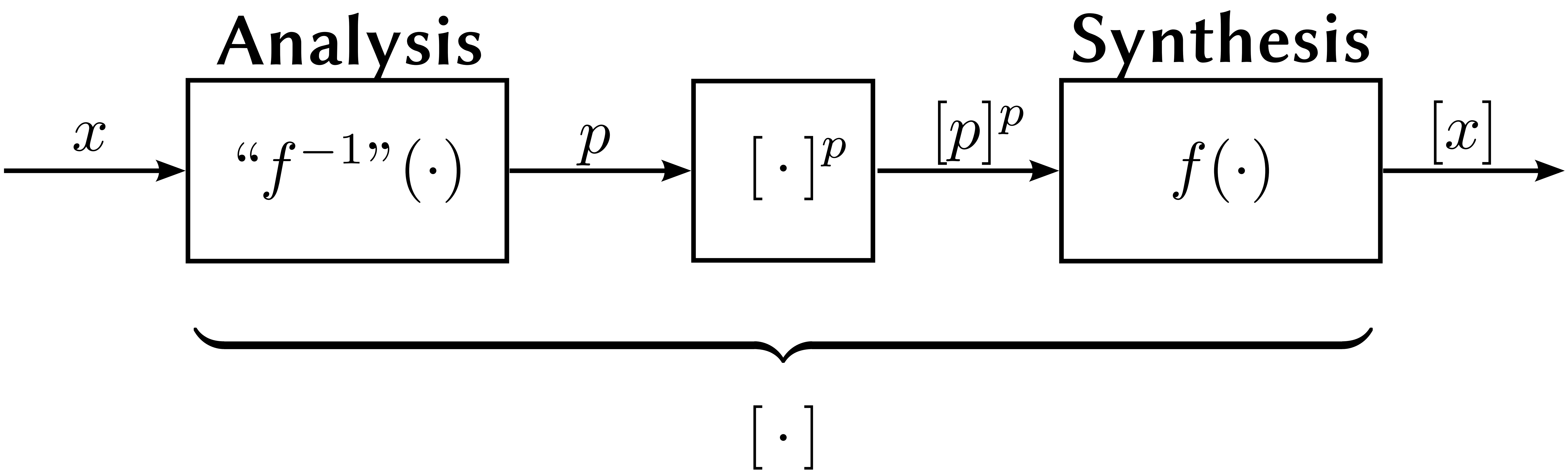
Analysis



$$p = \arg\min_{p'} \{ j(p') \mid f(p') = x \}$$

Parametric Models

Quantization



Linear Prediction Principles

Given a sequence of values $x_0, x_1, \cdots, x_{n-1}$, find the best approximation $\hat{x}_i$ of x_i such that:

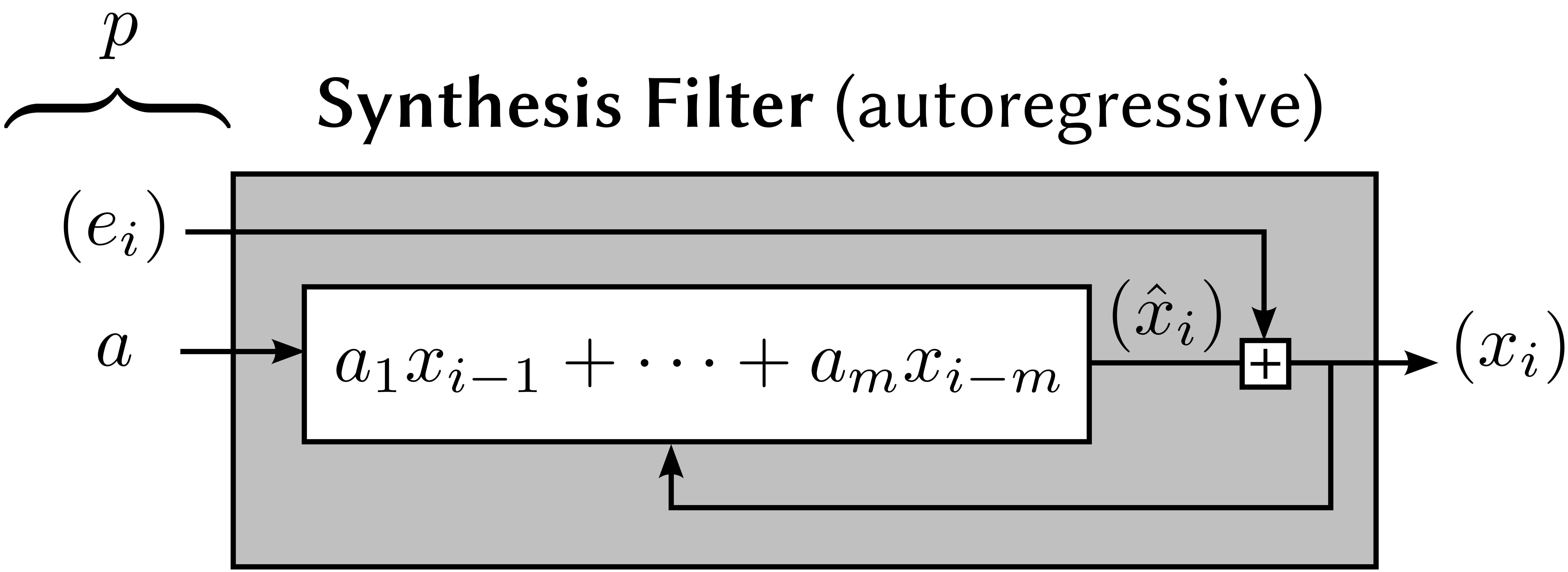
$$\hat{x}_i = a_1 x_{i-1} + a_2 x_{i-2} + \cdots + a_m x_{i-m}$$

The prediction is **linear** and (strictly) **causal**.
The number m is the **prediction order**.
The **prediction error/residual** is defined by:

$$e_i = x_i - \hat{x}_i$$

$$(x_i = \hat{x}_i + e_i)$$

Linear Prediction



Linear Prediction

Covariance Method

Solve

$$a^\star = \operatorname*{arg\,min}_{a \in \mathbb{R}^m} j(a)$$

with

$$j(a) = \sum_{i=m}^{n-1} (x_i - a_1 x_{i-1} - \cdots - a_m x_{i-m})^2$$

Linear Prediction

$$a^\star = \arg\min_{a \in \mathbb{R}^m} \|Aa - b\|^2$$

Covariance Method:

$$A = \begin{bmatrix} x_{m-1} & x_{m-2} & \cdots & x_0 \\ x_m & x_{m-1} & \cdots & x_1 \\ \vdots & \vdots & \vdots & \vdots \\ x_{n-2} & x_{n-3} & \cdots & x_{n-m-1} \end{bmatrix}, \; b = \begin{bmatrix} x_m \\ x_{m+1} \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Linear Prediction

$$a^\star = \arg\min_{a \in \mathbb{R}^m} \|Aa - b\|^2$$

Unique solution if A is into:

$$a^\star = [A^t A]^{-1} A^t b$$

Otherwise, one solution of:

$$j(a^\star) = \min_{a \in \mathbb{R}^n} j(a)$$

provided by:

$$a^\star = A^\sharp b \quad \text{where} \quad A^\sharp = \lim_{\epsilon \to 0} [A^t A + \epsilon I]^{-1} A^t$$

Linear Prediction

With numpy.linalg least-square solution to $Ax = b$:

```
def lp(x, m):
    "Linear predictor coefficients -- covariance method"
    n = len(x)
    A = array([x[m - 1 - arange(0, m) + i] for i in range(n-m)])
    b = x[m:n]
    a = lstsq(A, b)[0]
    return a
```

AutoCorrelation

Variant: treat the data as an infinite signal.

Set $x_i = 0$ if $i \neq 0, \cdots, n-1$ and minimize:

$$j(a) = \sum_{i=-\infty}^{+\infty} (x_i - a_1 x_{i-1} - \cdots a_m x_{i-m})^2$$

The solutions are the same if we minimize:

$$j(a) = \sum_{i=0}^{n+m-1} (x_i - a_1 x_{i-1} - \cdots a_m x_{i-m})^2$$

AutoCorrelation

(Implemented in the audio.lp module)

```
def lp(x, m, method="covariance"):
    "Linear predictor coefficients - cov./autocor. methods"
    if method == "autocorrelation":
        x = r_[zeros(m), x, zeros(m)]
    n = len(x)
    A = array([x[m - 1 - arange(0, m) + i] for i in range(n-m)])
    b = x[m:n]
    a = lstsq(A, b)[0]
    return a
```

Method Selection

Covariance:

- Fast computation,
- Accurate solution.

Autocorrelation:

- Even faster computation,
- Stable synthesis filter.

Infinite Order

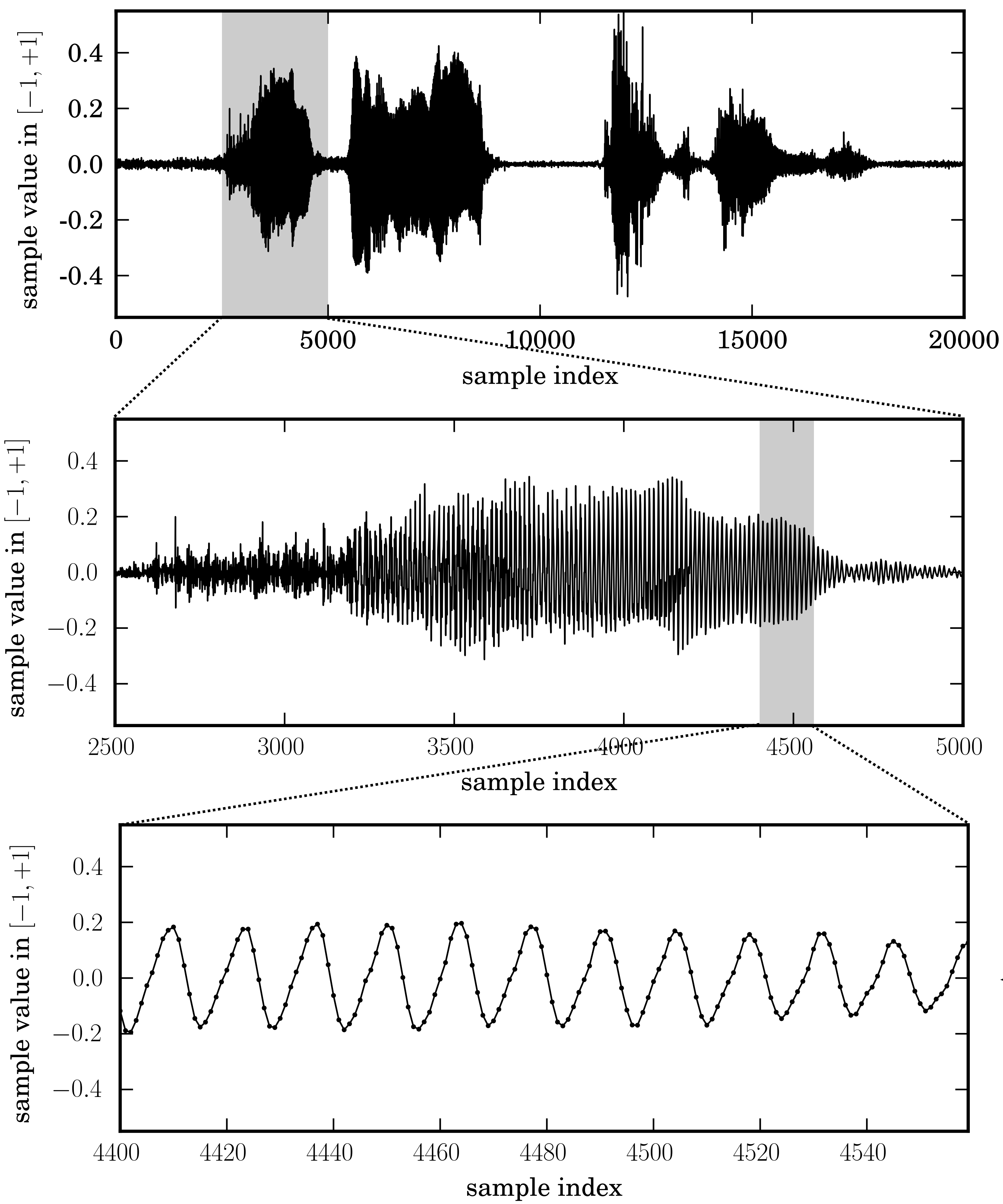
Define $e_i = x_i - \sum_{j=1}^{+\infty} a_j x_{i-j}$ for $a \in L^2(\mathbb{N}^*)$

If a is a minimum of $j(a) = \sum_{i=-\infty}^{+\infty} e_i^2$ then

$$\forall\, j \in \mathbb{Z}^*,\ \sum_{i \in \mathbb{Z}} e_i e_{i+j} = 0 \iff |e(f)| = \text{const.}$$

The prediction error is a **white noise.**

Voice Audio Data



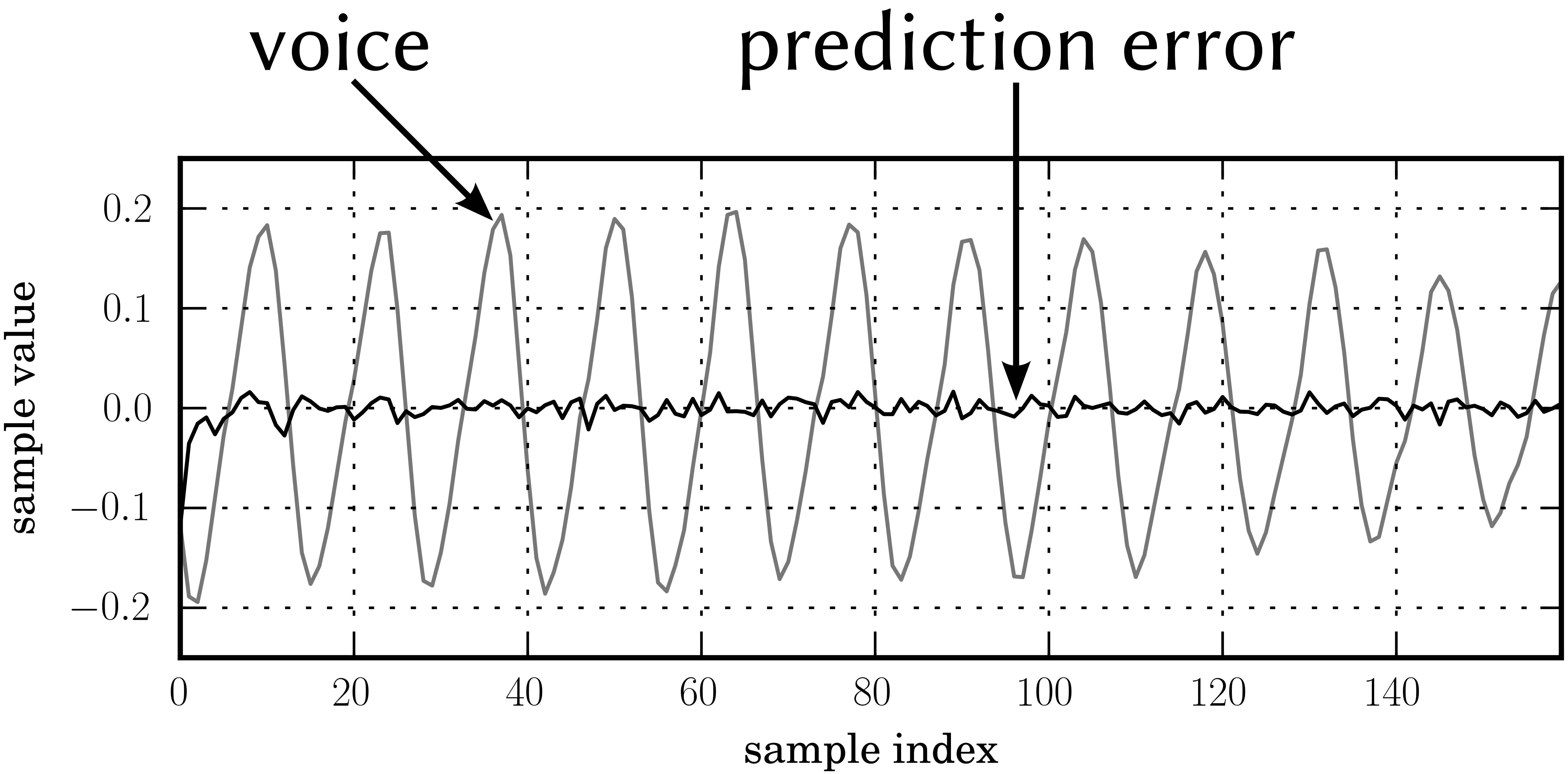
You Wanna Have Babies ?

8 kHz, mono.

20 ms frame

Voice Audio Data

Short-Term Prediction



order 16, autocorrelation

Analysis Filter

Finite Impulse Response filter - FIR

$$y_n = a_0 u_n + a_1 u_{n-1} + \cdots + a_{N-1} u_{n-N+1}$$


```
class FIR(Filter):
    def __call__(self, input):
        output = self.a[0] * input + dot(self.a[1:], self.state)
        self.state = r_[input, self.state[:-1]]
        return output
```

FIR Example

Moving Average

$$y_n = 0.25\left(u_n + u_{n-1} + u_{n-2} + u_{n-3}\right)$$

```
>>> from audio.filters import FIR
>>> ma = FIR(0.25 * ones(4))
>>> ma(1.0)
0.25
>>> ma(2.0)
0.75
>>> ma([3.0, 4.0])
array([1.5, 2.5])
```

FIR for Linear Prediction

```
>>> a = lp(data, order=m, ...)
```

Predictor

```
>>> predictor = FIR(r_[0, a])
```

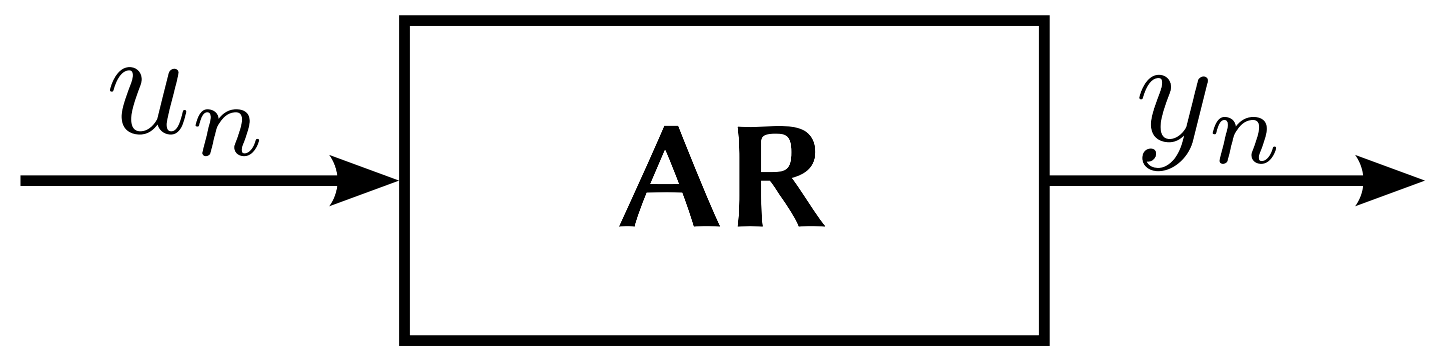
Analysis Filter

```
>>> error = FIR(r_[1.0, -a])
```

Synthesis Filter

Auto-Regressive filter - AR

$$y_n = a_1 y_{n-1} + \cdots + a_N y_{n-N} + u_n$$




```
class AR(Filter):
    def __call__(self, input):
        output = dot(self.a, self.state) + input
        self.state = r_[output, self.state[:-1]]
        return output
```

AR Example

$$y_n = 0.5 \times y_{n-1} + u_n$$

```
>>> from audio.filters import AR
>>> ar = AR([0.5])
>>> ar(1.0)
1.0
>>> ar(0.0)
0.5
>>> ar(0.0)
0.25
>>> ar([0.0, 0.0])
array([ 0.125 ,  0.0625])
```

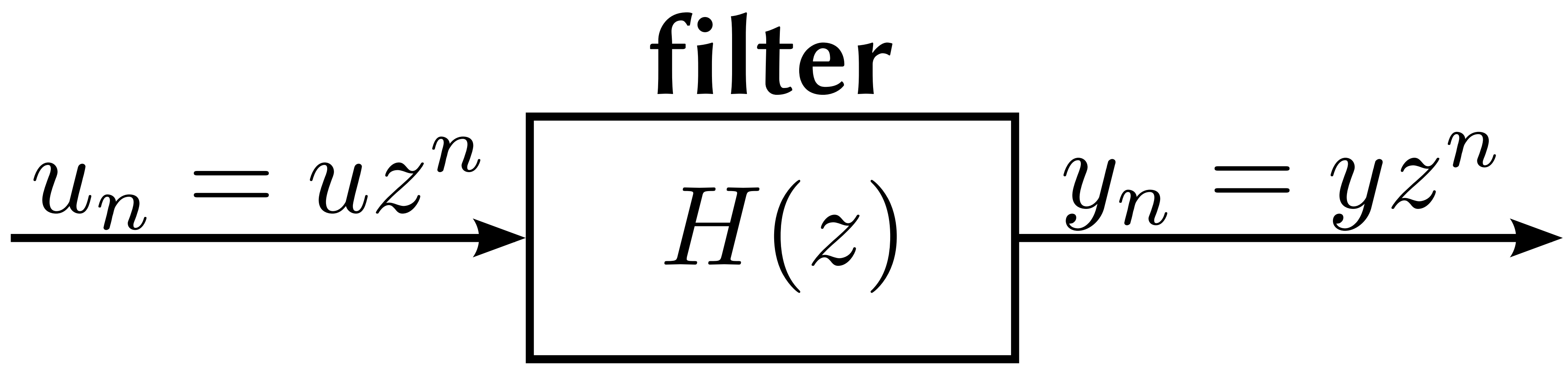
AR for Linear Prediction

Synthesis Filter

```
>>> a = lp(data, m, ...)
>>> synthesis = AR(a)
>>> data = synthesis(error)
```

Transfer Function

Given $z \in \mathbb{C}$, the inputs and outputs below



are related by:

$$y = H(z)u$$

FIR: $H(z) = a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{-N+1}$

AR: $H(z) = \dfrac{1}{1 - a_1 z^{-1} - \cdots - a_N z^{-N}}$

(I/O) Stability

A filter with a transfer function $H(z)$ is **stable** if:

- any bounded input yields a bounded output,
- the modulus of any pole $H(z)$ is less than 1.

Every FIR is stable, AR filters may be unstable ...

```
>>> ar = AR([1.0, 1.0, 1.0, 1.0])
>>> ar.poles()
array([ 1.92756198+0.j , -0.77480411+0.j ,
-0.07637893+0.81470365j, -0.07637893-0.81470365j])
>>> max(abs(pole) for pole in ar.poles())
1.9275619754829254
```

Frequency Response

It can be deduced from the transfer function:

$$H(f) = H(z = \exp(i2\pi f \Delta t))$$

The input signal

$$u(t) = Ae^{i(2\pi f t + \phi)}$$

generates the output

$$y(t) = H(f) \times Ae^{i(2\pi f t + \phi)}$$

or equivalently

$$y(t) = A' e^{i(2\pi f t + \phi')} \text{ with } \left| \begin{array}{l} A' = |H(f)| A \\ \phi' = \phi + \angle H(f) \end{array} \right.$$

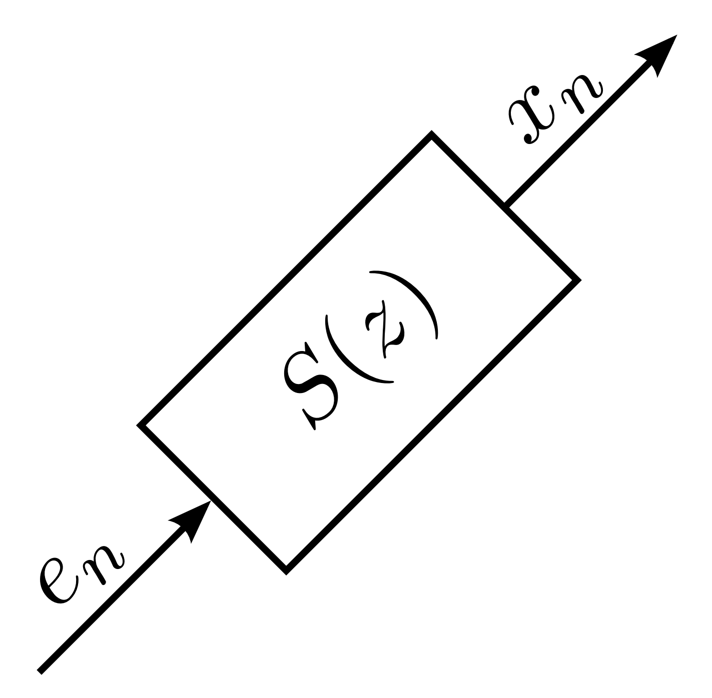
Spectral Analysis

A voice spectrum may be (locally) computed by a (windowed) FFT, or **periodogram**.

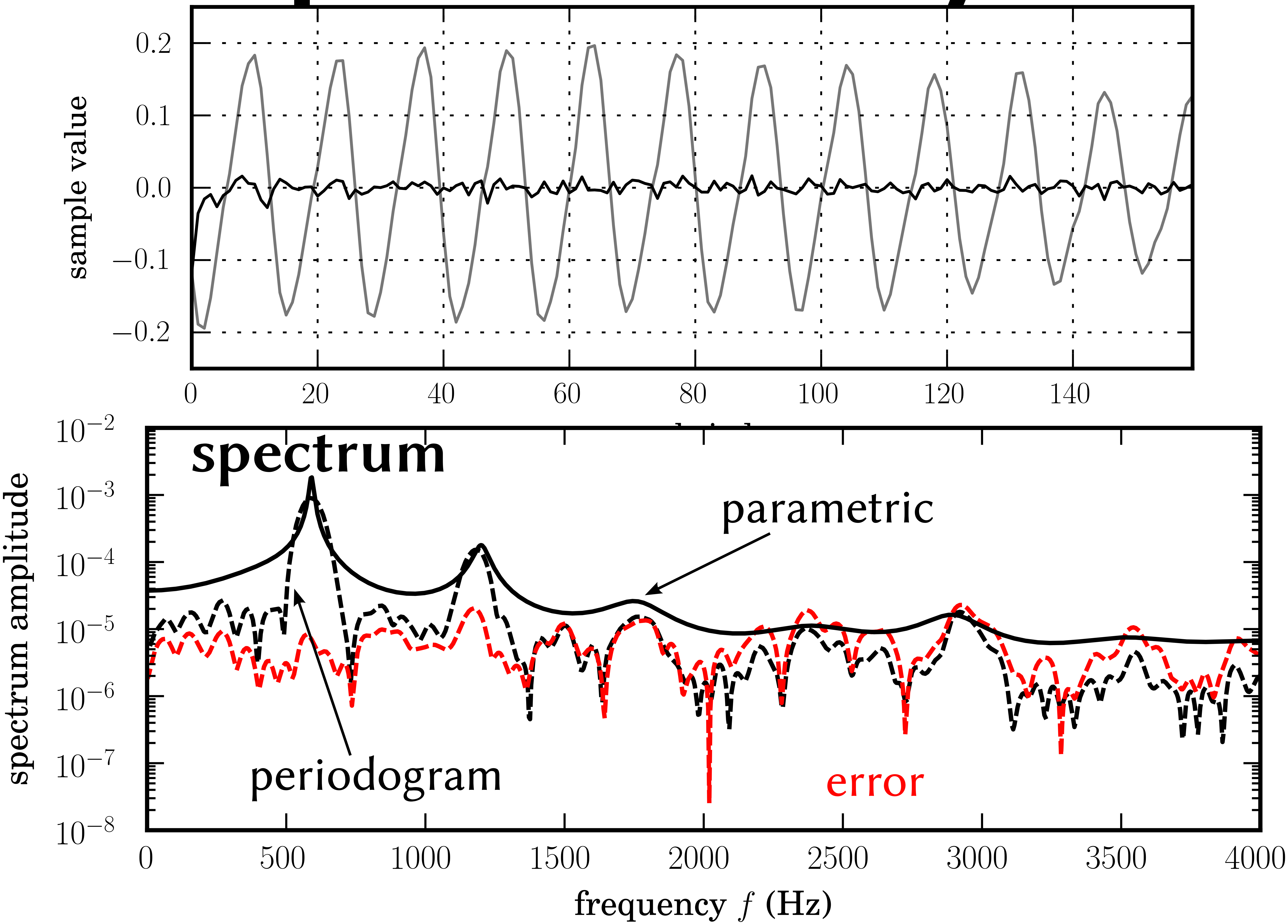
$$|x(f)| = \Delta t \left| \sum_{t \in \mathbb{Z}\Delta t} x(t) \exp(-i2\pi f t) \right|$$

If the prediction of this data has was successful, the error is (almost) white: $|e(f)| \simeq \text{const.}$, and the synthesis filter $S(z)$ provides a **parametric** spectrum estimate:

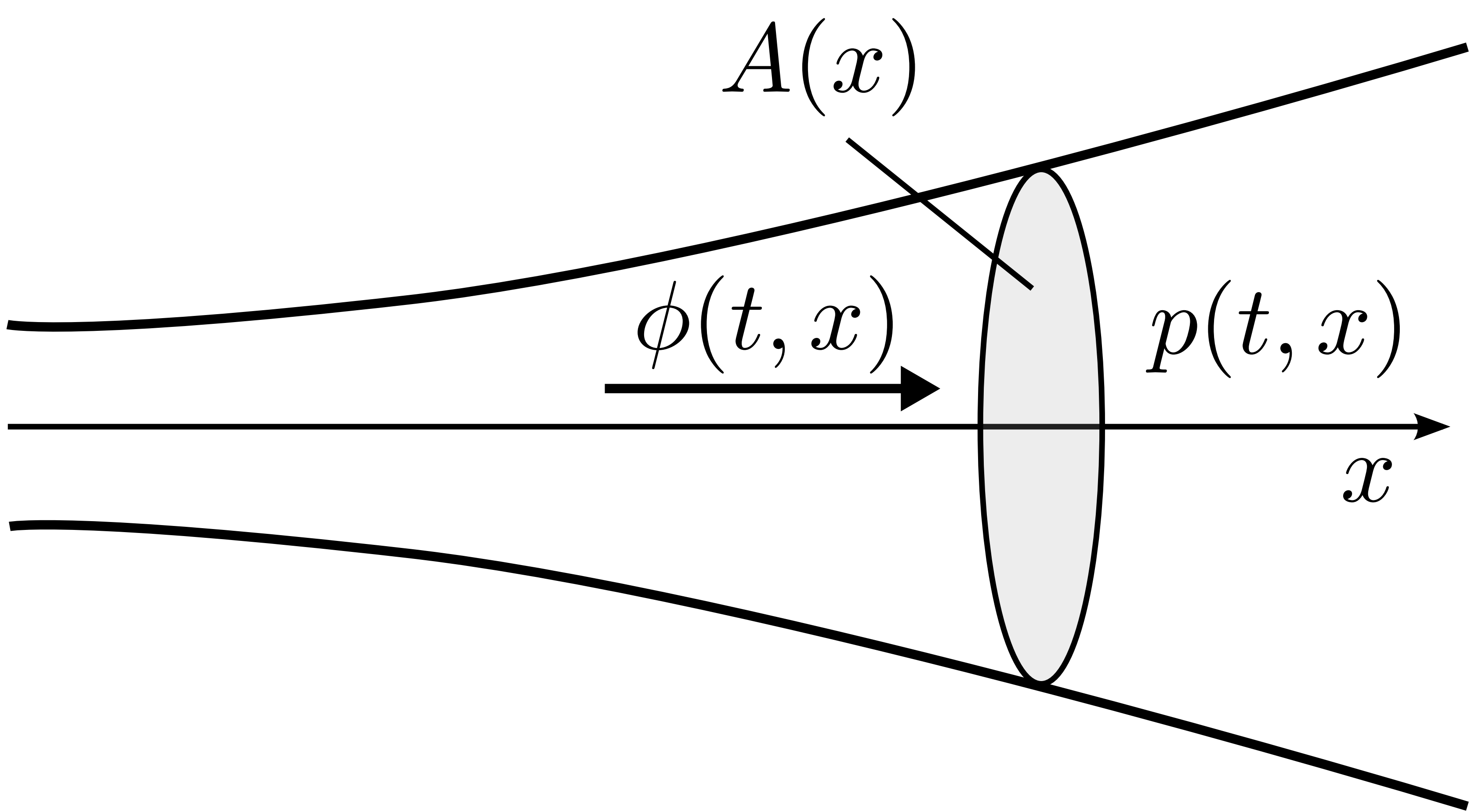
$$|x(f)| = |S(f)||e(f)| \propto |S(f)|$$



Spectral Analysis



Vocal Tract: Horn Model



A : cross-section

p : pressure

ϕ : air flow

Law of Motion

$$\rho \frac{\partial \phi}{\partial t} = -A \frac{\partial p}{\partial x}$$

density

Compressibility

$$K \frac{\partial \phi}{\partial x} = -A \frac{\partial p}{\partial t}$$

bulk modulus

Webster's Equation

Wave Velocity	Impedance
$c = \sqrt{\dfrac{K}{\rho}}$	$Z = \dfrac{\sqrt{K\rho}}{A}$

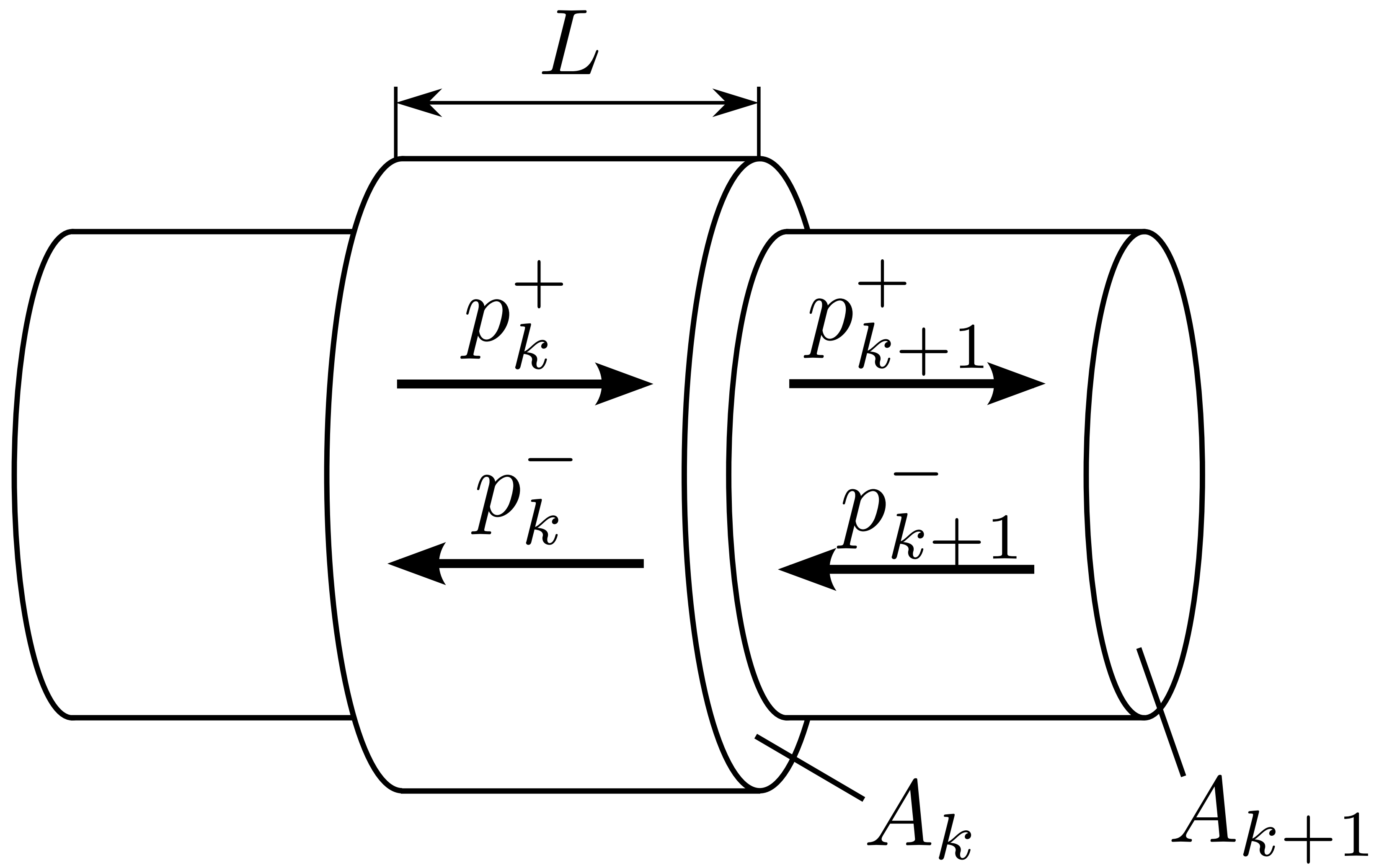
$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{A}\frac{dA}{dx}\frac{\partial p}{\partial x} - \frac{\partial^2 p}{\partial x^2} = 0$$

If A is constant:

$$\left| \begin{aligned} p(t,x) &= p^+(x-ct) - p^-(x+ct) \\ \phi(t,x) &= \phi^+(x-ct) - \phi^-(x+ct) \end{aligned} \right.$$

$$Z = \pm \frac{p^\pm}{\phi^\pm}$$

Discrete Model



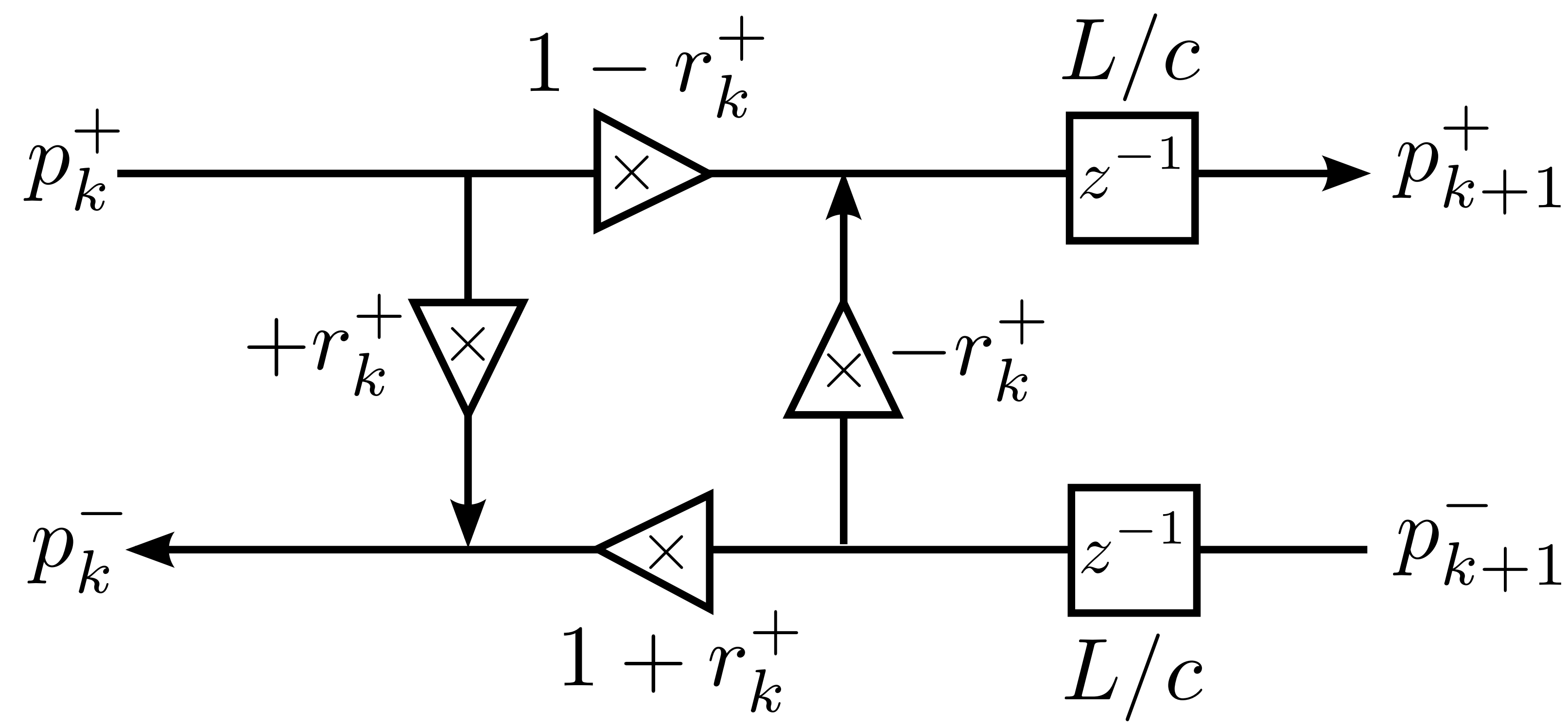
$$
\begin{aligned}
p_{k+1}^+(x-ct) &= (1-r_k^+)p_k^+(x-ct) + r_{k+1}^- p_{k+1}^-(x+ct) \\
p_k^-(x+ct) &= (1-r_{k+1}^-)p_{k+1}^-(x+ct) + r_k^+ p_k^+(x-ct)
\end{aligned}
$$

Reflection Coefficients:

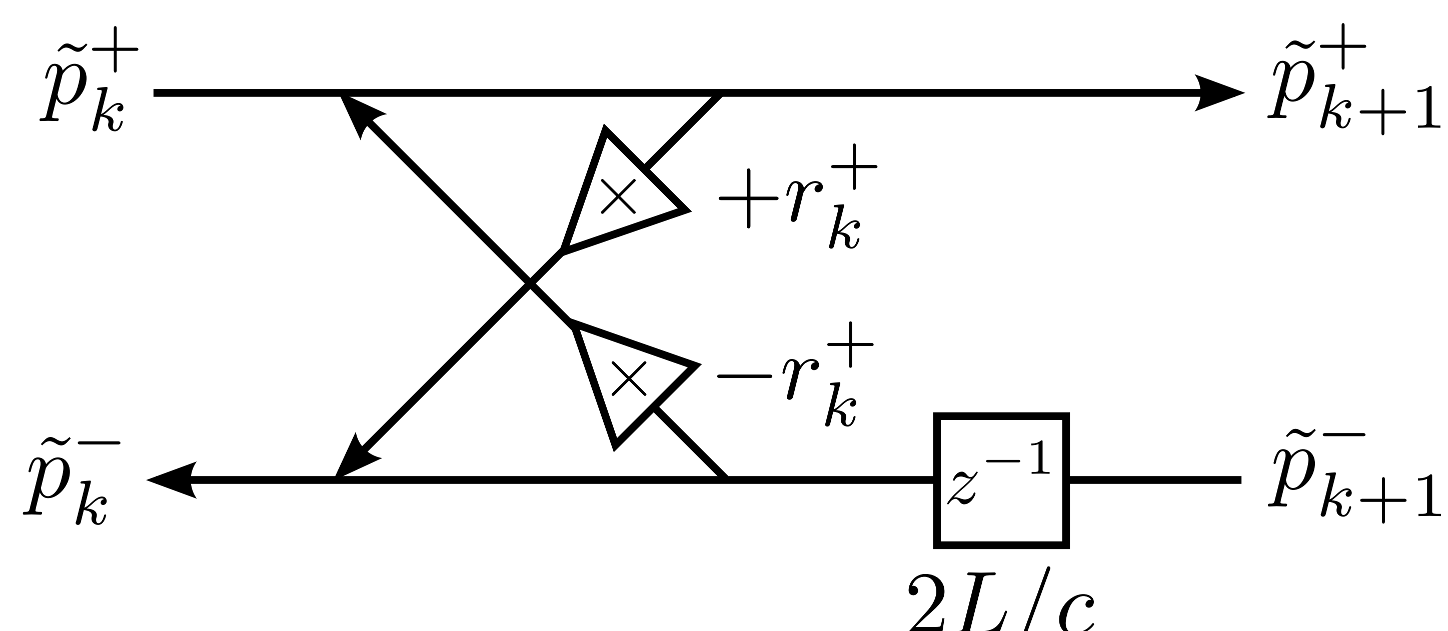
$$
r_k^+ = -r_{k+1}^- = \frac{A_{k+1} - A_k}{A_{k+1} + A_k}.
$$

Ladder/Lattice Filters

Kelly-Lochbaum Junction



Compensate p^+ for delay and attenuation:



Linear Predictive Coding

AR synthesis filters: lattice filters often replace register-based implementations.

- **Levinson-Durbin** and **Schur** algorithms are fast and provide directly the reflection coefficients that match the experimental data.

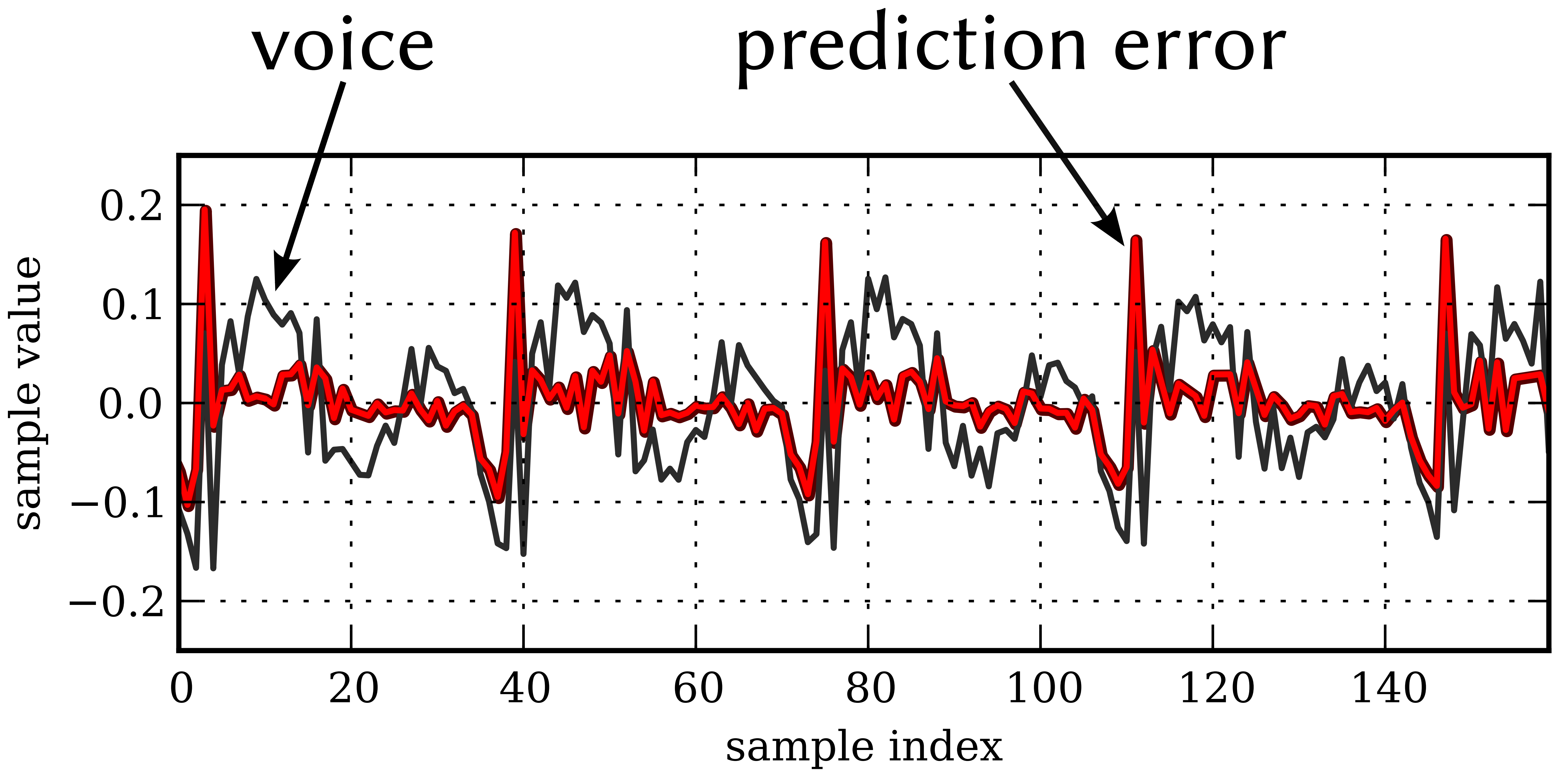
$$r_k^+ \longleftrightarrow a_i$$

- Synthesis filter are stable iff $|r_k^+| < 1$.
  Quantize the **area ratio** and preserve stability:

$$\frac{1 + r_k^+}{1 - r_k^+} = \frac{A_{k+1}}{A_k}$$

Voice Audio Data

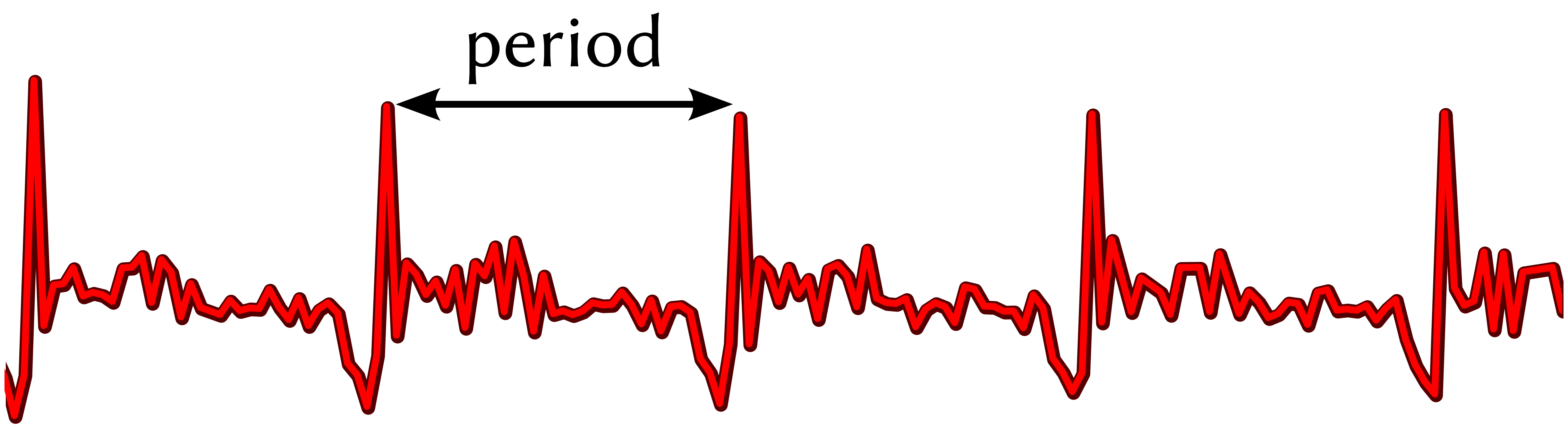
Short-Term Prediction



order 16, autocorrelation

Pitch

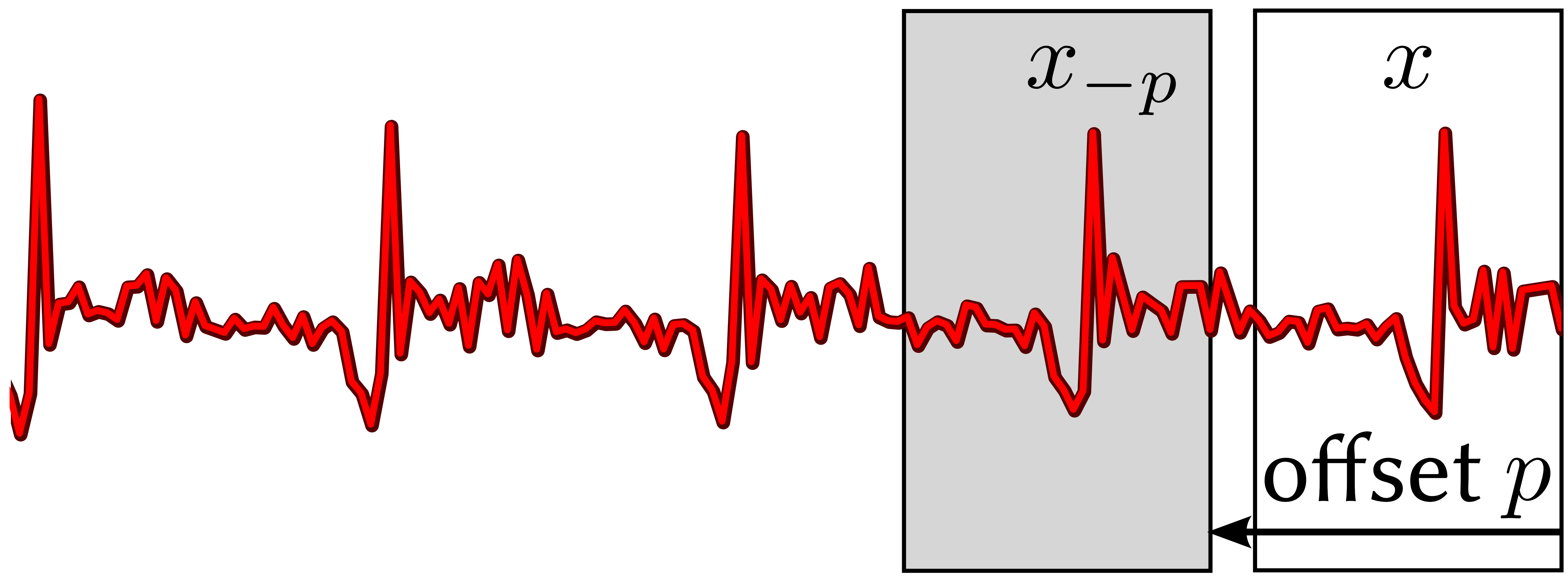
Voiced sounds are identified by prediction errors that are sequences of impulses.



The **pitch** frequency is the inverse of the (pseudo-)period.

Basic Pitch Detection

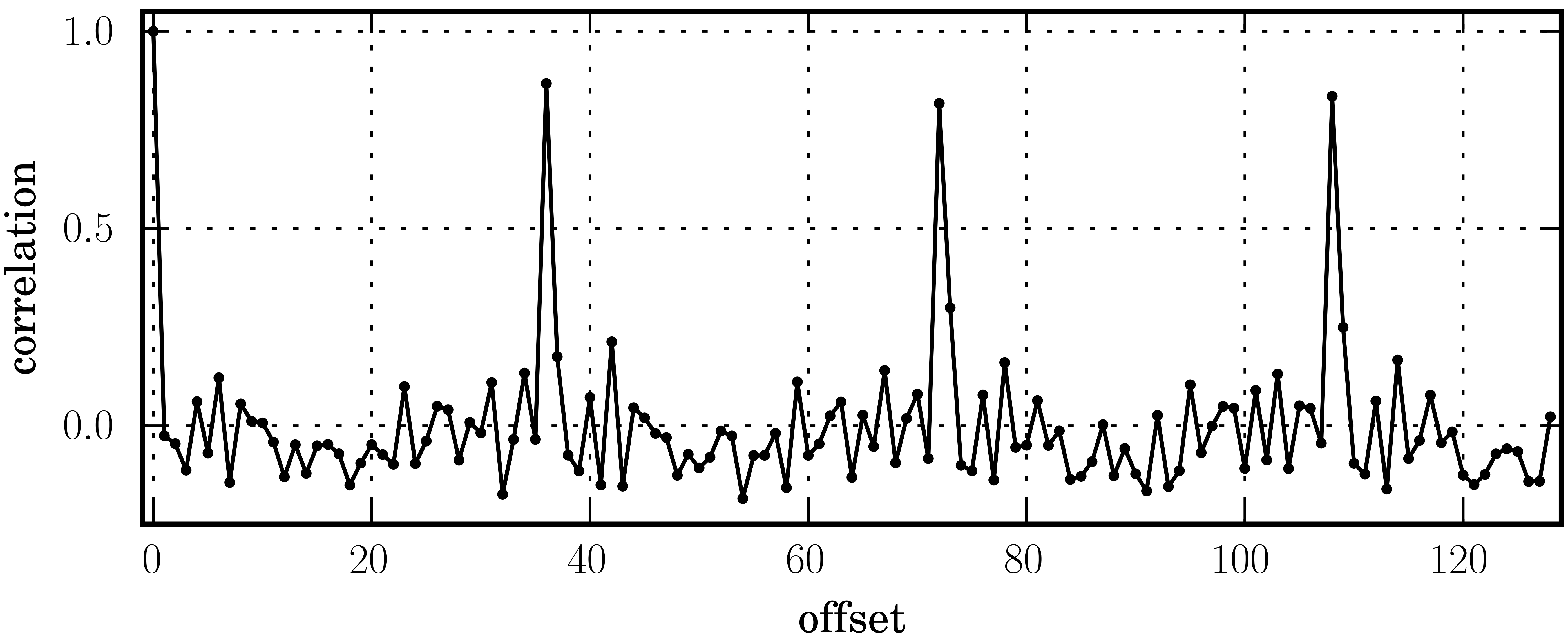
Solve $\min_{p \in \mathbb{N}^*} \min_{k \in \mathbb{R}} \|e(p,k)\|^2$ where $e(p,k) = x - kx_{-p}$



$$\frac{\|e(p,k^\star)\|^2}{\|x\|^2} = 1 - \left\langle \frac{x_{-p}}{\|x_{-p}\|}, \frac{x}{\|x\|} \right\rangle^2$$

$$k^\star = \frac{\langle x_{-p}, x \rangle}{\|x_{-p}\|^2} \qquad p^\star = \arg\max_{p \in \mathbb{N}^*} \left| \left\langle \frac{x_{-p}}{\|x_{-p}\|}, \frac{x}{\|x\|} \right\rangle \right|$$

Autocorrelation Function



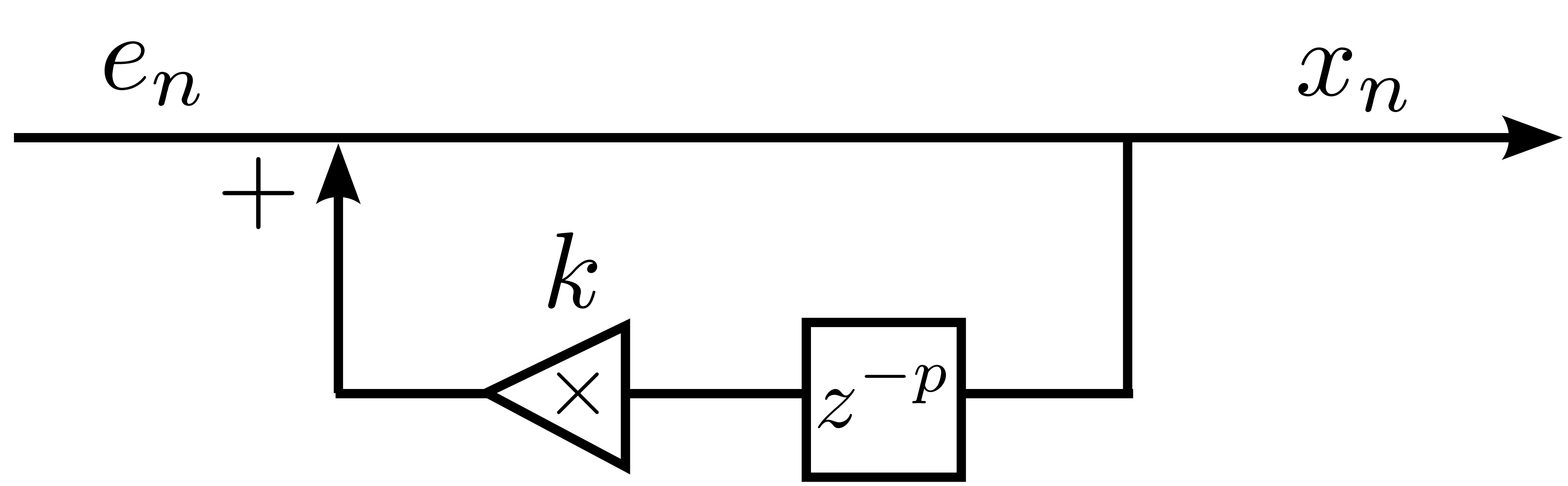

```
def ACF(data, frame_length):
    def normalize(x):
        return x / norm(x)
    m, n = frame_length, len(data)
    A = array([normalize(data[n-m-i:n-i]) for i in range(n-m+1)])
    return dot(A, normalize(data[-m:]))
```

Long-Term Prediction

LTP residual

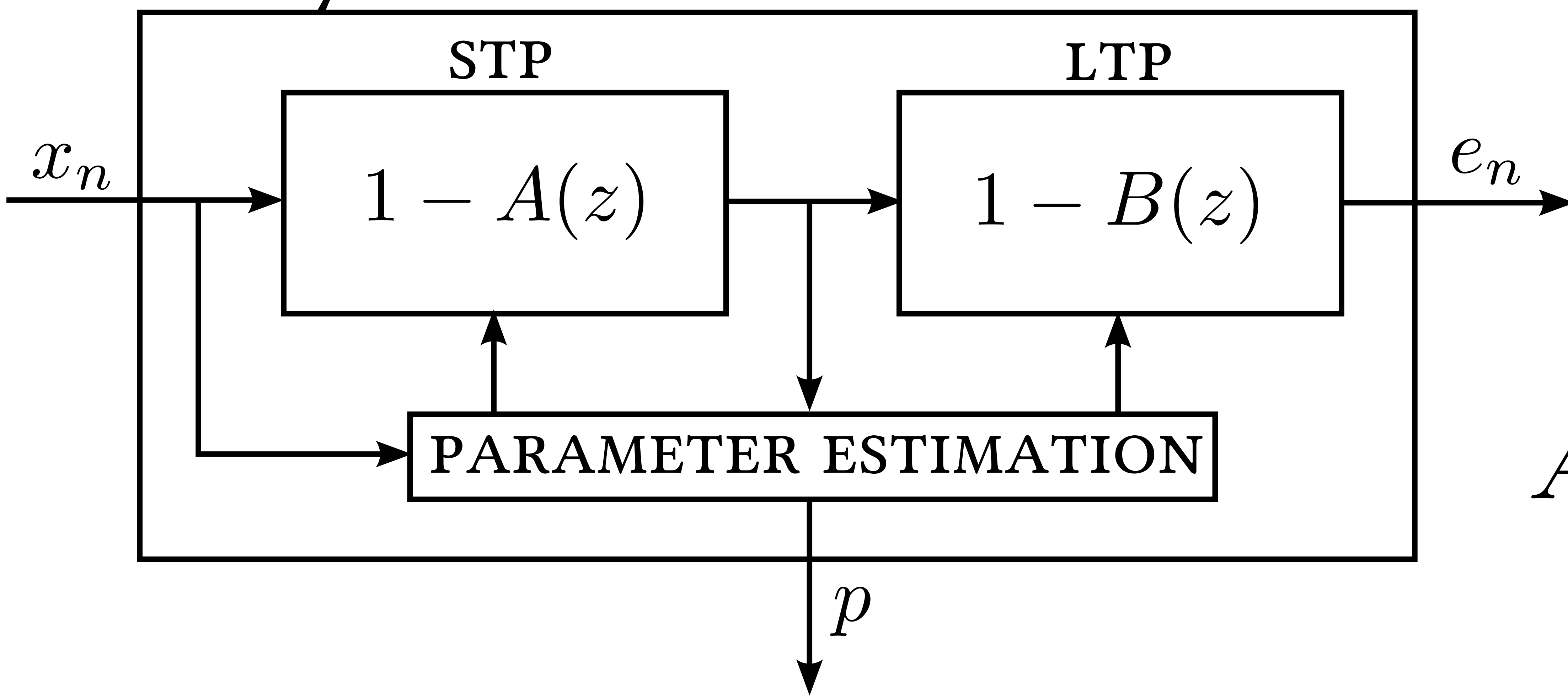
$$x_n = kx_{n-p} + e_n$$

LTP Synthesis (AR) Filter

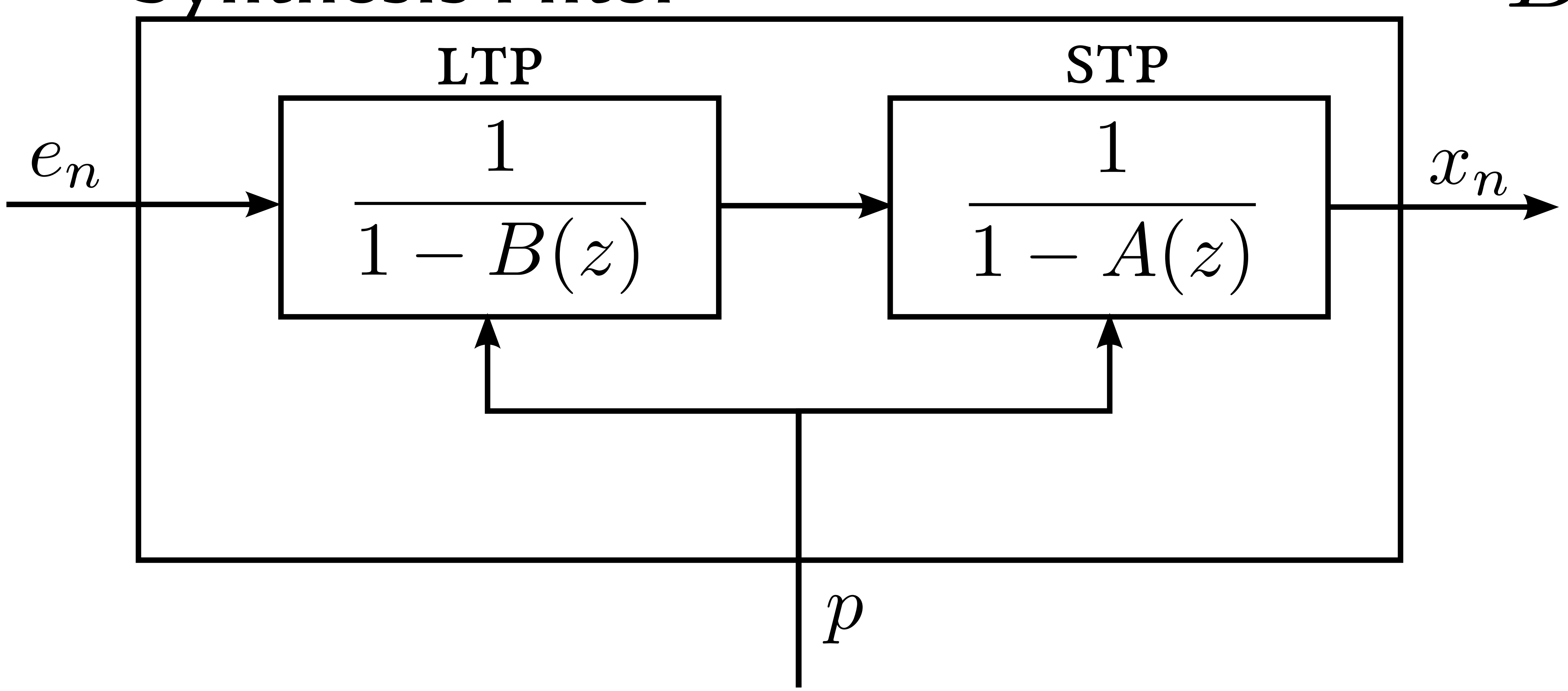


Linear Predictive Coding

Analysis Filter



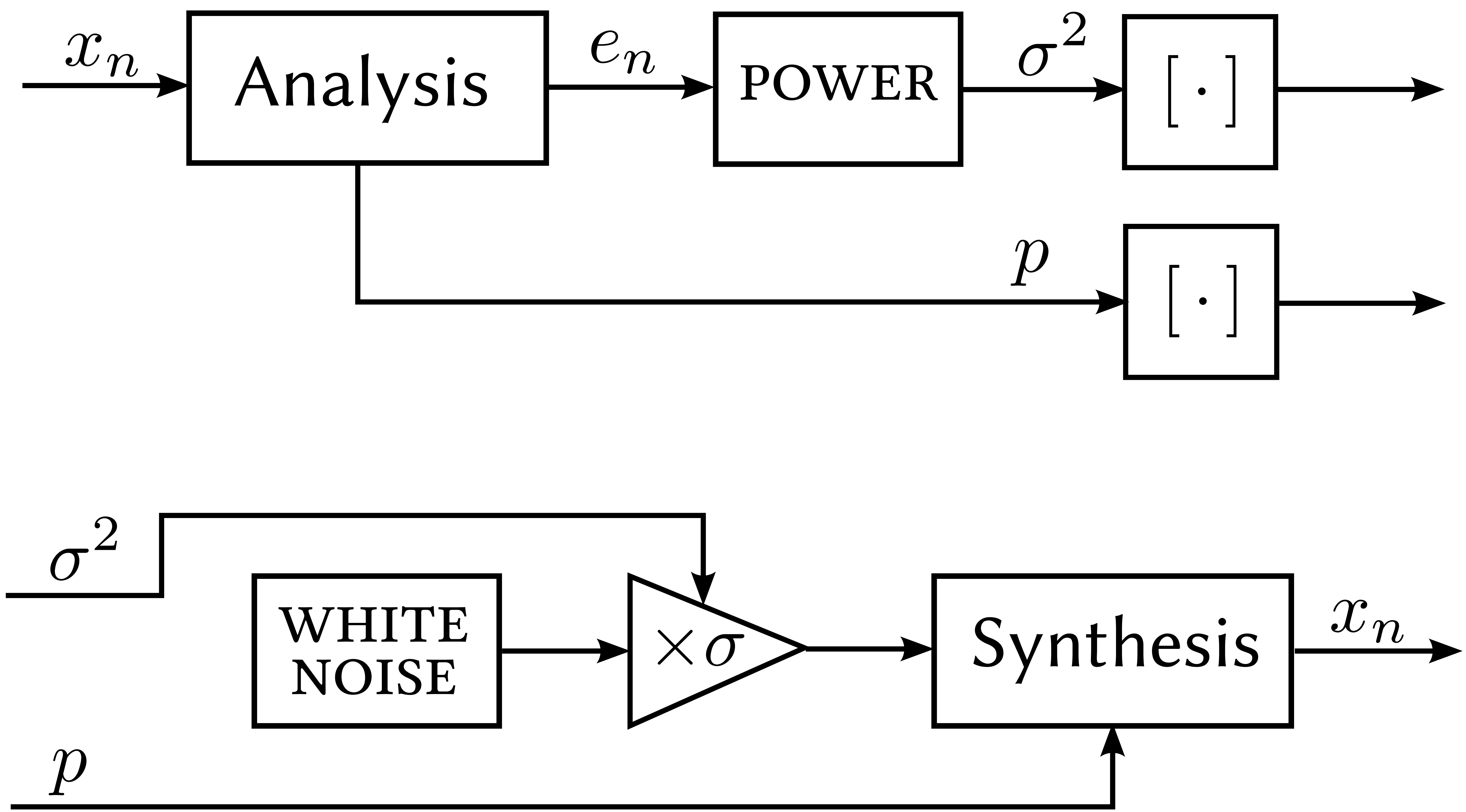
Synthesis Filter



$$A(z) = \sum_{i=1}^{m} a_i z^{-i}$$

$$B(z) = k z^{-p}$$

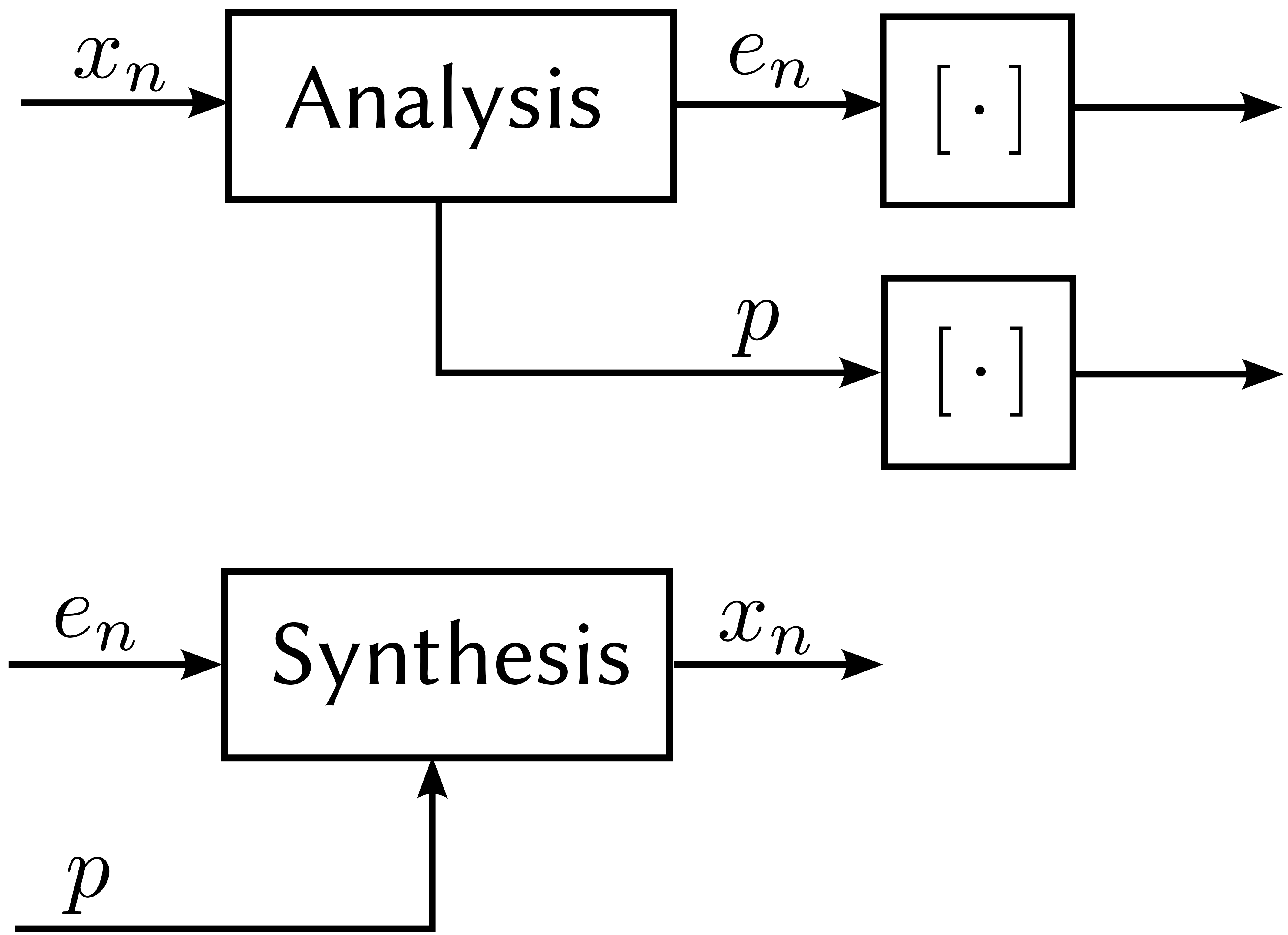
"Pure" LPC



x_n
Analysis
e_n
POWER
σ^2
$[\cdot]$
p
$[\cdot]$
σ^2
WHITE NOISE
$\times\sigma$
Synthesis
x_n
p

APC / RELP

Adaptative Predictive Coding
Residual-Excited Linear Prediction



CELP

Code-Excited Linear Prediction